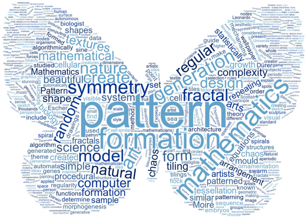
Advanced Mathematical Perspectives 1

Lecture 11: Diffusion in 2D



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Fourier's analytical theory of heat (final form, 1822), devised in the Galileo-Newton tradition of controlled observation plus mathematics, is the ultimate source of much modern work in the theory of functions of a real variable and in the critical examination of the foundation of mathematics.

Eric Temple Bell, The Development of Mathematics (1940) p. 165

I fart in your general direction.

French soldier, Monty Python and the Holy Grail

Section 1

Diffusion in higher dimensions

Diffusion

- Assume mixing takes time, even in a gas
- Start with a concentration of some substance
- It spreads, or *diffuses* through the medium
- Its a model for
  - various fluids and gases
  - the spread of heat through a solid
  - some problems in electronics
  - spread of disease
- Key ideas
  - no material is created or destroyed, only moved around
  - rate of movement depends on concentrations themselves

Diffusion in higher dimensions

Imagine a (thin) metal bar, being heated at one end

Assumptions

- We can easily extend the idea to 2D – think of heating a plate of metal
- In 3D, think of a gas diffusing from a point in a room

Diffusion

Now position **x** is a vector

- $u(\mathbf{x}, t)$ is the temperature (in Kelvins)
  - at point $x \in [0, L]^n$ along n-D space with sides L
  - at time $t \geq 0$
- $c =$ *specific heat*
  $=$ amount of heat needed to increase a unit mass by one degree
- $\rho =$ *density* (mass per unit length)
- $k =$ *thermal conductivity*
- $\alpha =$ *thermal diffusivity*

$$\alpha = \frac{k}{c\rho}$$

i.e., how easy it is for heat to diffuse across the medium

material	α
copper	111
wood	0.082

Diffusion in 2D

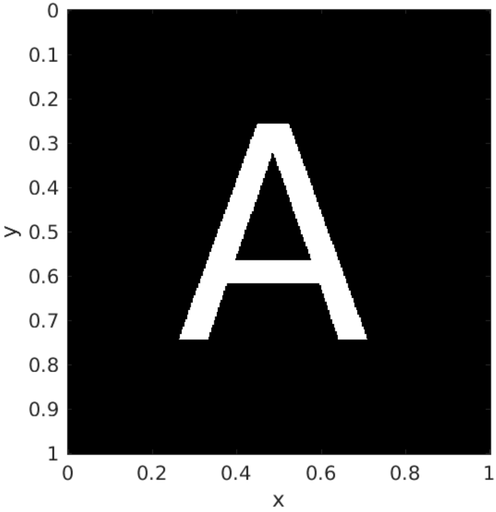
Generalise to 2D metal plate, temperature $u(x, y, t)$
Heat equation looks exactly the same:

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u$$

Laplacian becomes

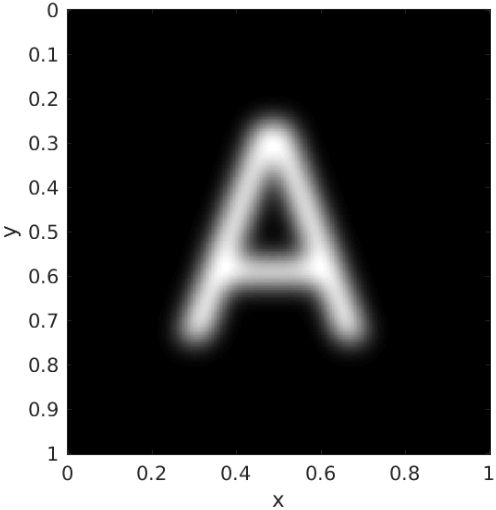
$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

Now the Laplacian encodes a direction, but it's all encapsulated in the same equation (which we could extend to 3D or more)



Original

www.maths.adelaide.edu.au/matthew.roughan/notes/AMP1/files/diffusion2.gif



At $t = 0.04$

www.maths.adelaide.edu.au/matthew.roughan/notes/AMP1/files/diffusion2.gif

Numerical Solution by Difference Equations

- 3D arrays $u(i, j, k)$
- Extend your derivatives to 2D equivalents, and iterate over 2 spatial dimensions, but otherwise everything is the same.

Takeaways

- Diffusion is one of the underlying models for many physical processes (often ones that build patterns)
- It results in "smoothing" of an initial signal, and this can be used in filtering and denoising patterns
- We have implicit filtering going on in our heads!
- We will come back to use diffusion again as part of a larger pattern formation process, but next we will look at another model for diffusion

Section 2

Analytic Solution of 1D Heat Equation

Separation of variables

We are looking for a solution to the heat equation

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u$$

Assume the solution can be written

$$u(x,t) = X(x)T(t),$$

i.e., the parts corresponding to the two variables separate.

Separation of variables

Assume $u(x,t) = X(x)T(t)$

$$\begin{aligned}\frac{\partial u}{\partial t} &= X(x)T'(t) \\ \frac{\partial^2 u}{\partial x^2} &= X''(x)T(t)\end{aligned}$$

Substitute into the heat equation, and divide by αXT, and we get

$$\frac{1}{\alpha}\frac{T'}{T} = \frac{X''}{X}$$

The LHS is a function of t only, and the RHS is a function of X only, so they must be equal to a constant, call it $-\lambda^2$, and then we can separate the equation into

$$T' + \lambda^2 \alpha T = 0 \quad (1)$$

$$X'' + \lambda^2 X = 0 \quad (2)$$

Separation of variables

The solutions of

$$T' + \lambda^2 \alpha T = 0 \quad (3)$$
$$X'' + \lambda^2 X = 0 \quad (4)$$

are

$$T(t) = t_0 e^{-\lambda^2 \alpha t}$$
$$X(t) = A \sin \lambda x + B \cos \lambda x$$

We can work out t_0, A, B, and λ from the initial and boundary conditions. It turns out there could be more than one λ_n involved, and we can get all of these from a Fourier transform of the initial state. So the final solution is given by a set of exponentially decaying sin and cosine functions (hence the connection to the previous lecture).

Further reading I