

The profound study of nature is the most fertile source of mathematical discoveries.

Joseph Fourier (1768-1830)

'I'll take spots, then,' said the Leopard; 'but don't make 'em too vulgar-big. I wouldn't look like Giraffe—not for ever so.'

*How the leopard got his spots, Just So Stories,
Rudyard Kipling*

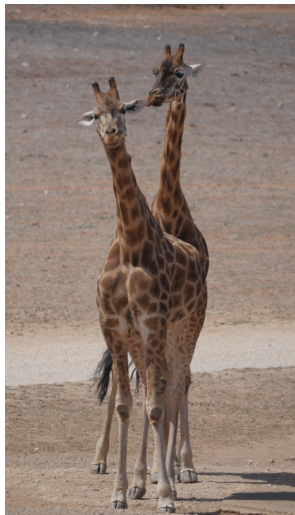
Section 1

Symmetry Breaking

Real Patterns

- Real patterns have periodicity
 - ▶ periodic patterns aren't perfectly symmetrical
 - ▶ e.g., they have n -fold rotational symmetry, not arbitrary rotational symmetry
- Real patterns have some broken symmetries

Real Patterns



Skin/hair patterns

- Patterns on mammals are usually from coloured hair
 - ▶ colour from pigments: melanin (eumelanin and phaeomelanin)
 - ▶ pigment from special cells: melanocytes
- Pigments produced by melanocytes depend on presence/absence of activator/inhibitor chemicals

Buridan's Ass

Ass = Donkey

- Donkey is placed *exactly* between two precisely *equal* stacks of hay.
- The donkey will go to the best or closest, but they are the same.
- So it starves to death because it can't decide which to go to.

Named after 14th century philosopher Jean Buridan, but idea goes back at least to Aristotle 350 BC.

Ex nihilo nihil fit: Nothing comes out of nothing

The idea underlying Buridan's Ass

- Take a system that has a particular symmetry
- Assume the laws of physics are symmetric, and that the system evolves under these laws
- The system cannot “lose” the symmetry
- But there are many cases that seem to contradict this.

Symmetry breaking in nature

- Most higher-life forms on our planet starts as a single *spherical* cell, but end up with only with bilateral symmetry
- Even on bilaterally symmetric animals, we see patterns that are not symmetric, *e.g.*, giraffe spots
- A flower starts as an (almost) cylindrical stem, but then creates petals, with only discrete rotational symmetry.

Primary question: how can symmetry break?

Secondary question: when the symmetry breaks, why does it do so in such a controlled way, *e.g.*, why are there usually the same number of petals?

Section 2

Reaction Systems

Growth equation

Something you should have seen before

$$\frac{dx}{dt} = kx,$$

Has solution

$$x(t) = x_0 \exp(kt).$$

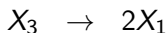
Describes:

- in finance, compound interest
- in biology, population growth
- in physics, chain reactions
- in computers, Moore's law
- ...

Reaction

Now imagine

- there are some *reagents*, and there may be some other chemicals presents that are involved but whose amounts don't change, e.g., catalysts
- the quantities of the reagents at time t are $u_i(t)$ for $i = 1, 2, \dots, n$
- there are some reactions that happen, described by a *stoichiometric* equation



we may not care about all outputs, and some inputs may be constantly refreshed (e.g., O_2 in air), so we don't need to keep track of everything.

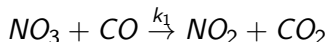
Law of Mass Action

The *rate* of a chemical reaction is directly proportional to the product of the concentrations of the reagents.

- Rate also has a constant of proportionality k
- Have to include a term for each stoichiometric term
- Presumes:
 - ▶ free mixing of reagents
 - ▶ elementary reaction (not one made up of multiple steps)

Reaction example [How09]

Nitrogen trioxide reacting with carbon monoxide to form nitrogen dioxide and carbon dioxide with rate k_1



Define

$u_1(t)$ = the quantity of NO_3

$u_2(t)$ = the quantity of CO

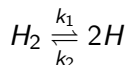
Equations

$$\begin{aligned}\frac{du_1}{dt} &= -k_1 u_1 u_2 \\ \frac{du_2}{dt} &= -k_1 u_1 u_2\end{aligned}$$

The amounts of NO_2 and CO_2 are implicit.

Reaction example [How09]

We can also have reactions that go in both directions



Take $u_1(t)$ = amount of H_2 and $u_2(t)$ = amount of H

Equations

$$\frac{du_1}{dt} = -k_1 u_1 + k_2 u_2^2$$

$$\frac{du_2}{dt} = +2k_1 u_1 - 2k_2 u_2^2$$

Autocatalysis

- A chemical reaction is *autocatalytic* if one of the reaction products is also a catalyst for the reaction.
- Classic example: Rabbits

Rabbits can make more rabbits, so the more rabbits you have ...

Activator-Inhibitor Systems

- One chemical is an *activator*, *i.e.*, it is autocatalytic, and so speeds up the reaction
- The other chemical is an *inhibitor*, *i.e.*, it slows the reaction down

Simple example: predator-prey model

- Rabbits autocatalyse (reproduce)
- Foxes eat the rabbits (but die if there isn't enough food)

Lotka-Volterra predator-prey model [Lot20]

$$\begin{aligned} \text{prey: } \frac{du_1}{dt} &= \alpha u_1 - \beta u_1 u_2 \\ \text{predator: } \frac{du_2}{dt} &= \delta u_1 u_2 - \gamma u_2 \end{aligned}$$

Constant parameters

α = breeding rate of rabbits (assumes plenty of food)

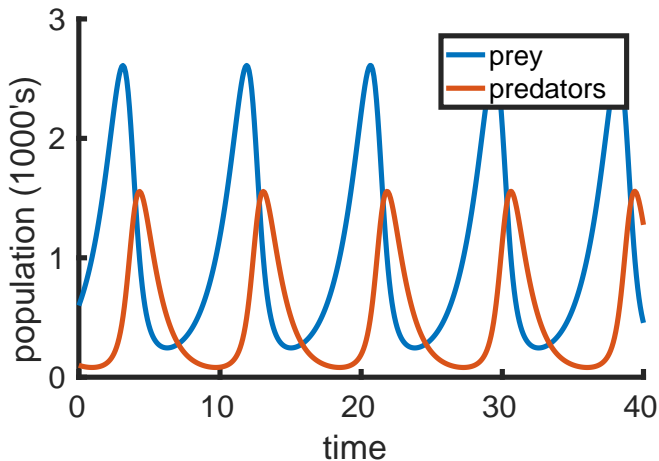
β = how quickly foxes eat rabbits

δ = death rate of foxes

γ = when foxes eat rabbits they breed (assumes foxes only eat rabbits)

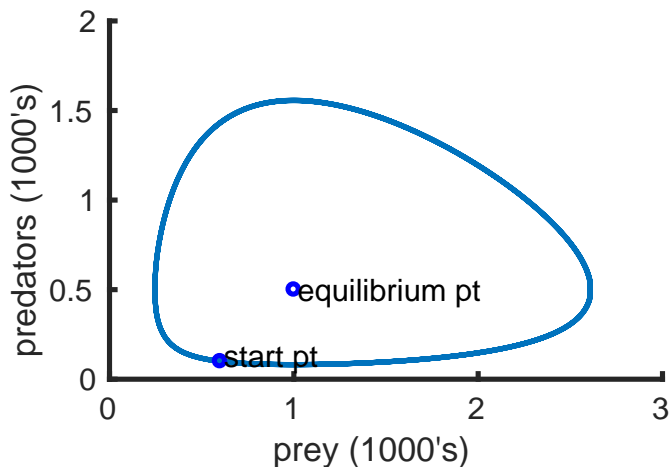
Lotka-Volterra predator-prey model

Solution to the equations are periodic, but aren't just a simple sinusoid



Lotka-Volterra predator-prey model

Often plot as a *phase* plot



Section 3

Stability and Instability

Equilibrium

(simple) equilibrium is where everything balances, so nothing changes, so

$$\frac{dx}{dt} = 0$$

e.g., Lotka-Volterra

$$\begin{aligned} \text{prey: } \frac{du_1}{dt} &= \alpha u_1 - \beta u_1 u_2 = 0 \\ \text{predator: } \frac{du_2}{dt} &= \delta u_1 u_2 - \gamma u_2 = 0 \end{aligned}$$

which has solutions $u_1 = u_2 = 0$ and

$$\begin{aligned} u_1 &= \frac{\gamma}{\delta} \\ u_2 &= \frac{\alpha}{\beta} \end{aligned}$$

Given these values, the system would remain static

Stability and instability

In real life, systems are subject to small, random *perturbations*

- When a *stable* system is perturbed (by a small amount) from equilibrium, it will return back
- When an *unstable* system is perturbed (even by a minuscule amount) the perturbation will be amplified, and we will drop away from equilibrium

Imagine a stick ...

Stability analysis is often performed by *linearising* the system, *i.e.*, finding a linear approximation to the system at equilibrium, because linear systems are easier to analyse.


Instability


- Perturbation from an unstable equilibrium can result in system eventually converging to a new (stable) equilibrium
- The symmetries of the two equilibriums might be different
- A *tiny* amount of noise in the perfect symmetry of the original equilibrium is enough to break it, but then it settles back to a regular (stable) equilibrium pattern with a different symmetry


Activities

- Projects!!!

Further reading I

 P. Howard, *Modeling with ODE*, 2009,
www.math.tamu.edu/~phoward/m442/modode.pdf.

 A.J. Lotka, *Analytical note on certain rhythmic relations in organic systems*, Proc Natl Acad Sci USA **6** (1920), no. 7, 410–415.

 Gudrun Scholz and Fritz Scholz, *First-order differential equations in chemistry*, ChemTexts **1** (2014), no. 1,
<https://link.springer.com/article/10.1007/s40828-014-0001-x>.