Information Theory and Networks

Lecture 19: Complexity

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Part I

Complexity

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Henry Louis Mencken

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Simplicity and Occam's razor

Pluralitas non est ponenda sine neccesitate
William of Ockham (ca. 1285-1349)

- "Plurality should not be posited without necessity."
- alternative versions
  - "Entia non sunt multiplicanda praeter necessitatem", or "Entities should not be multiplied beyond necessity"
  - "in vain we do by many which can be done by means of fewer"
  - "if two things are sufficient for the purpose of truth, it is superfluous to suppose another"
  - Principle of Parsimony

William of Ockham (ca. 1285-1349) was a medieval English philosopher and Franciscan monk. Franciscans were minimalists, idealising a life of poverty and simplicity. We can see that in his statement, which is perhaps aimed more at a philosophy of life and theology, than of science. It also isn't necessarily his idea, though we associate it with him through his writings.
http://skepdic.com/occam.html

Quidquid latine dictum sit, altum viditur.

Complexity

- Occam's Razor is often interpreted as "simple theories are best" (all else being equal)
- But what do "simple" or "complex" mean?
  - computational complexity
    - computational resources (e.g. CPU or memory) required by an algorithm
  - emergence and self-organization
    - e.g. flocking behaviour
    - e.g. Conway's game of life
    - e.g. consciousness
  - non-linearity and "chaos"
  - irreducible systems
    - systems that are more than the sum of their parts?
  - programming complexity
    - metrics for describing how complicated a computer program is
    - e.g., length of code, vocabulary,
    - e.g., count of linearly independent paths through the code

http://en.wikipedia.org/wiki/Complex_systems
http://en.wikipedia.org/wiki/Programming_complexity
http://en.wikipedia.org/wiki/Lehman's_laws_of_software_evolution

"complicated" vs "complex"

Warren Weaver, 1948

- **disorganised complexity:**
  - large number of relationships, often can be considered almost independent
  - "complicated" = lots of moving parts, but reducible to these
  - use probability and statistical mechanics to analyse, e.g., temperature of a gas, roll of a dice, ...
- **organised complexity:**
  - smaller (maybe still large) number of relationships, that can't be treated as independent
  - non-random, but hard to predict
  - "complex" = small number of parts can generate "interesting" behaviour
  - analyse (typically) through simulation

http://en.wikipedia.org/wiki/Complexity

Complexity

- Why do I care:
  - complex systems are harder to manage
  - how can we make them simpler if we don't even understand what that means
- We're interested in strings (signals or messages) so lets talk about them?

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Complexity examples

- We're interested in strings (signals or messages) so lets talk about them?
- Which of these is complex?
  1. 10101010101010101010101010101010
  2. 11001001000011111101101010100010 0
  3. 10100101010000101010111101010101 0

Complexity examples

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- Which of these is complex?
  1. 10101010101010101010101010101010
  2. 11001001000011111101101010100010 0
  3. 10100101010000101010111101010101 0
- Answers:
  1.
  2.
  3.

Kolmogorov Complexity

- The basic idea is that the complexity is the length of the shortest description of the sequence
  - "description" could mean a program to generate it
  - or it could just be "write the string 10101..."
- Obviously this is still a little vague
  - what programming language and computer?

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Turing Machine

- An abstract model of a computer
- Turns out that all sufficiently complex computing systems are equivalent in the sense that they can compute the same family of functions:
  - computable functions intuitively have a finite program, that completes in a finite number of steps to the result
  - almost all functions we deal with in math are computable (though maybe not efficiently)
  - there are a few that aren't
- Turing machines have a few variants, but simplest has
  - a tape
  - a finite state machine that can write/read from the tape

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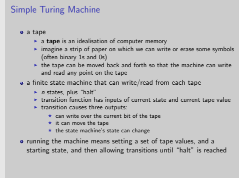
The Church-Turing thesis states, informally, that if some algorithm exists to compute a function, then the same calculation can be calculated on a Turing machine, or with Church's λ-calculus, or in fact any sufficiently complex computer.

Note, not all functions are computable. The Busy-Beaver function is an example that takes an input n, and returns the largest number of symbols that any Turing machine with n states can print before halting, when run with no input. It turns out not to be computable.

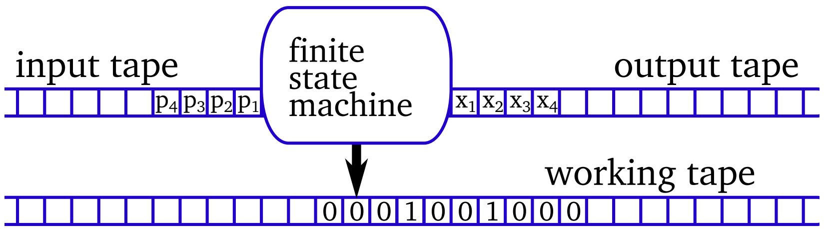
Simple Turing Machine

- a tape
  - a **tape** is an idealisation of computer memory
  - imagine a strip of paper on which we can write or erase some symbols (often binary 1s and 0s)
  - the tape can be moved back and forth so that the machine can write and read any point on the tape
- a finite state machine that can write/read from each tape
  - n states, plus "halt"
  - transition function has inputs of current state and current tape value
  - transition causes three outputs:
    - can write over the current bit of the tape
    - it can move the tape
    - the state machine's state can change
- running the machine means setting a set of tape values, and a starting state, and then allowing transitions until "halt" is reached

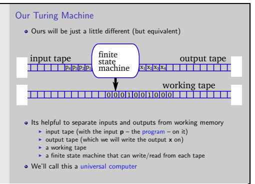


Our Turing Machine

- Ours will be just a little different (but equivalent)



- Its helpful to separate inputs and outputs from working memory
  - input tape (with the input **p** – the program – on it)
  - output tape (which we will write the output **x** on)
  - a working tape
  - a finite state machine that can write/read from each tape
- We'll call this a universal computer



Formal Kolmogorov Complexity

Definition (Kolmogorov Complexity)

The Kolmogorov complexity $K_{\mathcal{U}}(\mathbf{x})$ of a string $\mathbf{x}$ with respect to a universal computer $\mathcal{U}$ is defined as

$$K_{\mathcal{U}}(\mathbf{x}) = \min_{\{\mathbf{p} \mid \mathcal{U}(\mathbf{p})=\mathbf{x}\}} \ell(\mathbf{p})$$

So we are

- minising the length $\ell(\mathbf{p})$ of the input $\mathbf{p}$
- such that the output $\mathcal{U}(\mathbf{p}) = \mathbf{x}$
- and then it halts

A universal computer (or universal Turing machine) is one that can simulate any other Turing machine.

Further reading I

Thomas M. Cover and Joy A. Thomas, *Elements of information theory*, John Wiley and Sons, 1991.