THE UNIVERSITY of ADELAIDE

Examination in School of Mathematical Sciences
Semester 2, 2019

105637	**APP MTH 4052**	**Applied Mathematics Topic F: Complex Network Modelling and Inference**
105661	**APP MTH 7088**	**Applied Mathematics Topic F: Complex Network Modelling and Inference**

Official Reading Time:	10 mins
Writing Time:	180 mins
Total Duration:	190 mins

NUMBER OF QUESTIONS: 6 TOTAL MARKS: 50

Instructions

- Attempt all questions.
- Begin each answer on a new page.
- Examination materials must not be removed from the examination room.

Materials

- 1 Blue book is provided.
- Calculators without remote communications facilities are permitted.
- Students are permitted to bring two, double-sided pages of handwritten notes.
- English and foreign-language dictionaries may be used.

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO.

1. (a) Given the following adjacency matrix, draw an illustration of the corresponding graph G, labelling the nodes in the order implied by the matrix.

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) Is the graph G directed? Justify your answer.

(c) Is the graph G strongly connected? Justify your answer.

(d) Write the edge list of the graph G.

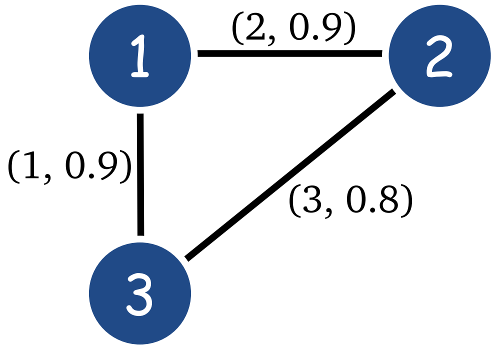
(e) Calculate the degrees of the nodes?

(f) Describe the handshake theorem, and its application to this graph.

[9 marks]

Please turn over for page 3

2. The following figure shows a network, with numbers giving the length of each link, followed by its reliability (the probability of packets successfully traversing the link).



   (a) Describe a semiring that could be used to define/solve the problem of finding the shortest paths, excepting that when there is a tie we choose the most reliable path. You do not have to prove that the algebra has the requisite properties. [Hint: it is possible to form algebras from tuples].

   (b) Write the generalised adjacency matrix for this network in this semiring.

   (c) Use the Floyd-Warshall algorithm to find the best paths between all pairs of nodes in this network. Show your working.

[9 marks]

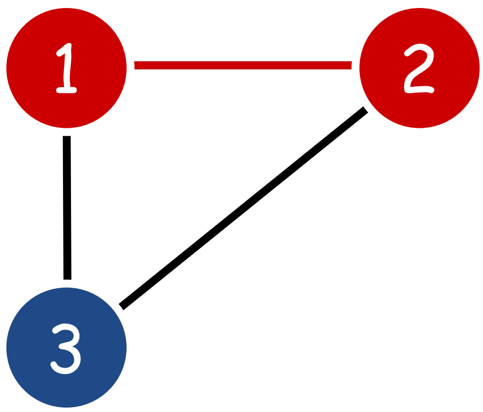
3. Consider the Gilbert-Erdős-Rényi random network $G(n, p)$ in the limit as $n \to \infty$ while $np = \lambda$, a constant greater than 1. Imagine that we want to sample from such a graph to measure its properties. N.B., all sampling is assumed to allow replacement.

   (a) If we sampled nodes uniformly at random where 1 in m nodes are sampled ($m \gg 1$), what node degree distribution should we observe in the **sampled subgraph**? Justify your answer.

   (b) If we sampled edges uniformly at random where 1 in k edges are sampled ($k \gg 1$), what node degree distribution from the sampled part of the **original graph** should we observe? Justify your answer.

   (c) If we use snowball sampling with a radius of one hop from the seed nodes and $\ell \ll n$ seed nodes, what node degree distribution should we observe in the **sampled subgraph**? Justify your answer.

   Hint: in the figure below, imagine that the red nodes and edge are sampled (by whatever sampling scheme is used).



   Then the node degrees of the sampled nodes in the **sampled subgraph**, which includes only sampled nodes and edges, are 1 and 1 (for nodes 1 and 2) whereas the degrees of the sampled nodes in the sampled part of the **original graph** are 2 and 2 (also for nodes 1 and 2).

[10 marks]

4. Prove that if the semiring $(S, \oplus, \otimes, \bar{0}, \bar{1})$ is q-stable then for all $a \in S$ there is an a^* which solves the equation

$$a^* = (a \otimes a^*) \oplus \bar{1}.$$

and that (at least one) a^* satisfies

$$a^* = a^{(q)},$$

where $a^0 = \bar{1}$, $a^k = a \otimes a^{k-1}$, $a^{(q)} = 1 \oplus a \oplus a^2 \oplus \cdots \oplus a^q$.

[Hint: there are two parts to prove here.]

[6 marks]

Please turn over for page 6

5. (a) Define a breadth-first search algorithm.

[4 marks]

(b) Given an undirected graph $G = (N, E)$ describe an algorithm that runs in time $O(|E|)$ that checks whether G is a tree.

[2 marks]

(c) Give (in brief) an example of where we might use a de Brujin graph, and what algorithm we would most likely be using on this graph.

[2 marks]

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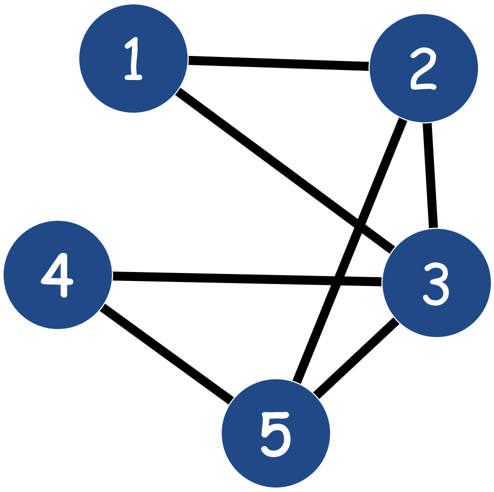
6. For the graph pictured below:



(a) (i) Find a (proper) node 3-colouring.

(ii) If there is a 2-colouring find it, or explain why it doesn't exist.

(iii) What is the chromatic number?

(iv) How many 3-colourings are there?

(v) The chromatic polynomial $P(G, t)$ is defined to be the polynomial that counts the number of t-colourings of a graph.

There are 96 4-colourings.

Write an expression for this graph's chromatic polynomial.

(b) Find the square-root graph.

(c) Is the graph planar? Explain.

[8 marks]