

# Optimisation and Operations Research

## Lecture 20: Primal Dual Algorithm, and Matrix Analysis of Simplex

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# Section 1

## Generalisations of Duals

# A reflection on duals

So why do we care about the dual and not just solve the primal?

- 1 It is sometimes easier to solve D (and they have the same optimal objective function value).
- 2 The complementary slackness theorem is very powerful/useful.
- 3 It is sometimes useful to consider both P and D together.
- 4 It helps in developing algorithms – primal algorithms, dual algorithms and primal-dual algorithms.

# Most generalised Primal/Dual

	Primal	Dual
1.	$\max z = \sum_{j=1}^n c_j x_j + z_0$	$\min w = \sum_{i=1}^m y_i b_i + z_0$
2.	$\sum_{j=1}^n a_{ij} x_j = b_i$	$y_i \text{ free}$
3.	$\sum_{j=1}^n a_{ij} x_j \leq b_i$	$y_i \geq 0$
4.	$\sum_{j=1}^n a_{ij} x_j \geq b_i$	$y_i \leq 0$
5.	$x_j \geq 0$	$\sum_{i=1}^m y_i a_{ij} \geq c_j$
6.	$x_j \leq 0$	$\sum_{i=1}^m y_i a_{ij} \leq c_j$
7.	$x_j \text{ free}$	$\sum_{i=1}^m y_i a_{ij} = c_j$

## Generalised duals and CSRs

At optimal solutions  $\mathbf{x}$  for  $(P)$  and  $\mathbf{y}$  for  $(D)$ , the following relationships must hold:

(a)  $x_j (\sum_{i=1}^m y_i a_{ij} - c_j) = 0, \quad j = 1, \dots, n.$

(b)  $(b_i - \sum_{j=1}^n a_{ij} x_j) y_i = 0, \quad i = 1, \dots, m.$

(a) is redundant if  $\sum_{i=1}^m y_i a_{ij} = c_j$  in  $(D)$

(b) is redundant if  $\sum_{j=1}^n a_{ij} x_j = b_i$  in  $(P)$  (LP has equalities),

otherwise this applies because  $s_i = b_i - \sum_{j=1}^n a_{ij} x_j$ . Hence this CSR is for the column corresponding to the slack variable  $s_i$  which has the associated dual constraint  $y_i \geq 0$  and therefore,  $s_i y_i = 0$ .

# Generalised duals and CSRs

## Example

Consider the following primal LP ( $P$ ).

$$\begin{aligned}(P) \quad \max z = & x_1 + 2x_2 + x_3 \\ & 2x_1 + x_2 \leq 6 \\ & -x_1 + x_2 + 2x_3 \leq 4 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.\end{aligned}$$

The dual ( $D$ ) is then

$$\begin{aligned}(D) \quad \min w = & 6y_1 + 4y_2 \\ & 2y_1 - y_2 \geq 1 \\ & y_1 + y_2 \geq 2 \\ & 2y_2 \geq 1 \\ & y_1 \geq 0, y_2 \geq 0.\end{aligned}$$

# Generalised duals and CSRs

## Example ((cont))

The CSR are

$$(2y_1 - y_2 - 1)x_1 = 0$$

$$(y_1 + y_2 - 2)x_2 = 0$$

$$(2y_2 - 1)x_3 = 0$$

$$(6 - 2x_1 - x_2)y_1 = 0$$

$$(4 + x_1 - x_2 - 2x_3)y_2 = 0.$$

If we found  $\mathbf{x} = (x_1, x_2, x_3)$  and  $\mathbf{y} = (y_1, y_2)$  which satisfied these equations and were feasible for  $(P)$  and  $(D)$ , respectively, what can we say about  $\mathbf{x}$  and  $\mathbf{y}$ ?

# Interpretation of the dual

Consider **Our First Problem** and its dual:

(P)

$$\max z = 13x_1 + 12x_2 + 17x_3$$

$$2x_1 + x_2 + 2x_3 \leq 225$$

$$x_1 + x_2 + x_3 \leq 117$$

$$3x_1 + 3x_2 + 4x_3 \leq 420$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

$$x_3 \geq 0$$

(D)

$$\min w = 225y_1 + 117y_2 + 420y_3$$

(labour)  $y_1 \geq 0$

(metal)  $y_2 \geq 0$

(wood)  $y_3 \geq 0$

(desks)  $2y_1 + y_2 + 3y_3 \geq 13$

(chairs)  $y_1 + y_2 + 3y_3 \geq 12$

(bedframes)  $2y_1 + y_2 + 4y_3 \geq 17$

**Economic interpretation:** First, note that

$y_1$  corresponds to a constraint about the usage of labour,

$y_2$  to a constraint on metal, and

$y_3$  to one on wood.

That is, each dual variable corresponds to a “cost” for using a resource.

The remaining three constraints then correspond to the costs for the items: desks, chairs, and bedframes.



## Interpretation of the dual (cont)

Suppose a speculator wants to purchase all of the manufacturer's resources.

Then the speculator must determine the price she/he is willing to pay for those resources.

Let

$y_1$  = price paid for one unit of labour (in \$)

$y_2$  = price paid for one unit of metal (in \$)

$y_3$  = price paid for one unit of wood (in \$).

The total cost of the resources is then

$$225y_1 + 117y_2 + 420y_3$$

and obviously, the speculator wishes to minimise this and hence we have the natural objective function in  $(D)$  above.

## Interpretation of the dual (cont)

Now consider the constraints on the speculator:

The speculator must pay the manufacturer an amount that is at least as big as the profit that the manufacturer can make on an item if the manufacturer is to be enticed to sell.

For example, consider the desk, where it takes 2 units of labour, 1 unit of metal and 3 units of wood to produce 1 desk, with a profit of \$13, the speculator must be willing to pay the manufacturer enough to cover the \$13; that is,  $2y_1 + y_2 + 3y_3 \geq 13$ . Similarly, we obtain constraints on chairs and bedframes. The nonnegativity constraints are obvious in this context.

In summary, when the primal is a maximisation problem, the dual variables are related to the value of the resources available to the decision maker.

Because of this, the dual variables are often referred to as *shadow prices*.

## Section 2

# Primal and dual algorithms

# Primal and dual algorithms

Recall that,

$$\left. \begin{array}{l} \text{primal feasibility} \\ \text{and dual feasibility} \\ \text{and complementary slackness} \end{array} \right\} \implies \text{Optimality}$$

or symbolically

$$\left. \begin{array}{l} Ax = b : x \geq 0 \\ \text{and } (\mathbf{y}^T A - \mathbf{c}^T) \geq 0 \\ \text{and } (\mathbf{y}^T A - \mathbf{c}^T)\mathbf{x} = 0 \end{array} \right\} \implies \text{Optimality}$$

## Primal and dual algorithms – 2

### **Primal algorithms:**

These maintain primal feasibility and complementary slackness and seek dual feasibility.

### **Dual algorithms:**

These maintain dual feasibility and complementary slackness and seek primal feasibility.

The dual simplex algorithm starts with and maintains a primal/dual basic solution that is dual feasible and satisfies complementary slackness while seeking primal feasibility.

# Dual simplex method – 1

## Example (Motivating Example)

Consider the problem:

$$\begin{aligned}\min z &= 2x_1 + 3x_2 + 4x_3 \\ x_1 + 2x_2 + x_3 &\geq 3 \\ 2x_1 - x_2 - 3x_3 &\geq 4 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.\end{aligned}$$

We convert to a maximisation problem and introduce slack variables

$$\begin{aligned}\max w &= -2x_1 - 3x_2 - 4x_3 \\ x_1 + 2x_2 + x_3 - x_4 &= 3 \\ 2x_1 - x_2 - 3x_3 - x_5 &= 4 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0.\end{aligned}$$

## Dual simplex method – 2

The associated tableau is

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	<b>b</b>
1	2	1	-1	0	3
2	-1	-3	0	-1	4
2	3	4	0	0	0

It can be seen that this is not in feasible canonical form

(multiplying the first two rows by  $-1$  results in the tableau being in canonical form, but not feasible canonical form as the **b** entries are negative).

We would therefore use Simplex Phase I to put the tableau in canonical form, and then apply Simplex Phase II to find the optimal solution.

## Dual simplex method – 3

Multiplying the first two rows by  $-1$  results in the tableau

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	<b>b</b>
$-1$	$-2$	$-1$	$1$	$0$	$-3$
$-2$	$1$	$3$	$0$	$1$	$-4$
$2$	$3$	$4$	$0$	$0$	$0$

It can be seen that  $\mathbf{y}^T = (0, 0)$  is a dual feasible solution: That is,  $\mathbf{y}^T A_N - \mathbf{c}_N^T = (2, 3, 4) \geq 0$  for the current basis  $B = \{x_4, x_5\}$ , but it is not primal feasible as  $x_4 = -3, x_5 = -4$ .

Just as in the primal simplex method we performed G–J pivot steps to retain primal feasibility and complementary slackness and sought dual feasibility, we may apply similar G–J pivot steps to retain the dual feasibility and complementary slackness and seek primal feasibility.



## Dual simplex method – 4

The steps for the dual simplex method are identical to the primal simplex method with the exception of choosing the pivot.

In the dual simplex method we select

- the *upmost* row with  $b_i < 0$ , and
- then select from the columns in that row from the set

$$N = \{j \mid a_{ij} < 0\} \text{ with minimum ratio } \frac{c_k}{a_{ik}}$$

- ▶ the leftmost column if there is a tie.

# Dual simplex method – 5

The tableau is

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	<b>b</b>
-1	-2	-1	1	0	-3
-2	1	3	0	1	-4
2	3	4	0	0	0
$\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{3}{2}$
$-\frac{5}{2}$	0	$\frac{5}{2}$	$\frac{1}{2}$	1	$-\frac{11}{2}$
$\frac{1}{2}$	0	$\frac{5}{2}$	$\frac{3}{2}$	0	$-\frac{9}{2}$
0	1	1	$-\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$
1	0	-1	$-\frac{1}{5}$	$-\frac{2}{5}$	$\frac{11}{5}$
0	0	3	$\frac{8}{5}$	$\frac{1}{5}$	$-\frac{28}{5}$

## Dual simplex method – 6

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	<b>b</b>
0	1	1	$-\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$
1	0	-1	$-\frac{1}{5}$	$-\frac{2}{5}$	$\frac{11}{5}$
0	0	3	$\frac{8}{5}$	$\frac{1}{5}$	$-\frac{28}{5}$

This is the final tableau as the solution is primal feasible, from which we can see the optimal solution is

$$x_1^* = 11/5, x_2^* = 2/5, x_3^* = 0 = x_4^* = x_5^*, \text{ with optimal value } z^* = -\frac{28}{5}$$

# Why dual algorithms

- Sometimes it's much easier to solve a dual problem
  - ▶ constraints might entangle lots of variables in the primal, but disentangle into small subproblems in the dual
  - ▶ example TCP
- Sometimes we are not solving LPs and it's harder to know when we have an optimal solution

## Section 3

# Matrix View of Simplex

# Matrix algebra and LP – 1

We first need to revise some results for solutions to a system of linear equations given by  $A\mathbf{x} = \mathbf{b}$ :

- 1 If  $A$  is a square  $m \times m$  invertible matrix and  $\mathbf{b} \in \mathbb{R}^m$ , then  $A\mathbf{x} = \mathbf{b}$  has a unique solution given by  $\mathbf{x} = A^{-1}\mathbf{b}$ .
- 2 If  $A$  is a non-square  $m \times n$  matrix, where  $n > m$ , with no row of  $A$  being a linear combination of any other rows (*i.e.*,  $\text{rank}(A) = m$ ), then there are:
  - 1 infinitely many solutions to  $A\mathbf{x} = \mathbf{b}$ , and
  - 2 infinitely many solutions to  $A\mathbf{y} = 0$ .

## Matrix algebra and LP – 2

It will be convenient to separate the basic and non-basic variables, by splitting the matrix  $A$  accordingly.

For example, given

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

and assuming  $B = \{x_1, x_2\}$  are the basic variables, and  $N = \{x_3\}$  the non-basic variable, we can rewrite the equation as

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} x_3 = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

or

$$A_B \mathbf{x}_B + A_N \mathbf{x}_N = \mathbf{b}.$$

## Matrix algebra and LP – 3

The column vector  $\mathbf{x}_B$  contains the variables in  $B$  and the column vector  $\mathbf{x}_N$  contains the variable(s) in  $N$ ,  $A_B$  has the columns of  $A$  corresponding to the variables in  $\mathbf{x}_B$  and  $A_N$  has the columns of  $A$  corresponding to the variables in  $\mathbf{x}_N$ .

If  $B$  has  $m$  elements and  $N$  has  $n - m$  (zero) elements, we set  $\mathbf{x}_N = \mathbf{0}$  and solve  $A_B \mathbf{x}_B = \mathbf{b}$  to find the basic solution  $\mathbf{x}_B = A_B^{-1} \mathbf{b}$ ,  $\mathbf{x}_N = \mathbf{0}$ .

We'll also make the following assumptions:

- 1 The matrix  $A$  has linearly independent rows (*i.e.*,  $\text{rank}(A) = m$ )
- 2 Any  $m \times m$  matrix formed from  $m$  columns of  $A$  is non-singular (*i.e.*, any set of  $m$  columns from  $A$  is linearly independent)
- 3 All basic solutions to  $\mathbf{x}_B = A_B^{-1} \mathbf{b}$ ,  $\mathbf{x}_N = \mathbf{0}$  have exactly  $m$  non-zero components (*i.e.*,  $x_i \neq 0$  for  $i \in B$ ).

Remarks: It is possible to solve LPs when  $A$  does not satisfy all of these assumptions. They are only made to simplify the theory in this section.



## Matrix algebra and LP – 4

Consider the standard equality form of an LP:

$$\begin{aligned} \text{(P)} \quad \max \quad z &= \mathbf{c}^T \mathbf{x} + z_0 \\ \text{s.t.} \quad A\mathbf{x} &= \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

where  $A = (a_{ij})$  is an  $m \times n$  matrix.

This problem is represented by the Simplex Tableau

$$M = \left[ \begin{array}{c|c|c} A & \mathbf{0} & \mathbf{b} \\ \hline -\mathbf{c}^T & 1 & z_0 \end{array} \right] \begin{array}{l} \} m \text{ rows} \\ \} 1 \text{ row} \end{array}$$

## Matrix algebra and LP – 5

Suppose the Simplex Method is applied to (P), and in the optimal solution, the set of basic variables is  $B = \{x_{j_k}, k = 1 \dots m\}$ .

For ease of reference, let us suppose

$$M = \left[ \begin{array}{c|c|c|c} A_B & A_N & \mathbf{0} & \mathbf{b} \\ \hline -\mathbf{c}_B^T & -\mathbf{c}_N^T & 1 & z_0 \end{array} \right] \begin{array}{l} \} m \text{ rows} \\ \} 1 \text{ row} \end{array}$$

$\begin{bmatrix} A_B \\ -\mathbf{c}_B^T \end{bmatrix}$  is the submatrix from the original tableau  $M$ , made up of columns corresponding to  $B$  (the basic variables).

$\begin{bmatrix} A_N \\ -\mathbf{c}_N^T \end{bmatrix}$  corresponds to columns for all the remaining (non-basic) variables.

# Matrix algebra and LP – 6

$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$z$	<b>b</b>
1	0	$\frac{3}{2}$	0	$\frac{1}{2}$	0	2
0	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	2
0	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0
0	0	3	0	2	1	6

With  $B = \{x_1, x_2, x_4\}$  and  $N = \{x_3, x_5\}$ , we re-arrange to the form

$x_1$	$x_2$	$x_4$	$x_3$	$x_5$	$z$	<b>b</b>
1	0	0	$\frac{3}{2}$	$\frac{1}{2}$	0	2
0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	2
0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0
0	0	0	3	2	1	6

## Matrix algebra and LP – 7

So without loss of generality, we can assume that  $M$  is partitioned as above, with

$$A = [A_B \ A_N]$$

as described.

Now all pivot steps in the simplex method are just Gauss-Jordan row operations and in the final tableau, the columns corresponding to the basic variables  $B$  are basic columns.

So we have

$$M = \left[ \begin{array}{c|c|c|c} A_B & A_N & \mathbf{0} & \mathbf{b} \\ \hline -\mathbf{c}_B^T & -\mathbf{c}_N^T & 1 & z_0 \end{array} \right] \xrightarrow{\text{G-J}} \hat{M} = \left[ \begin{array}{c|c|c|c} I & \hat{A}_N & \mathbf{0} & \hat{\mathbf{b}} \\ \hline \mathbf{0}^T & -\hat{\mathbf{c}}_N^T & 1 & \hat{z}_0 \end{array} \right].$$

## Matrix algebra and LP – 8

Recall from Maths I that a sequence of Gauss-Jordan row operations on a matrix  $M$  corresponds to premultiplying  $M$  by a set of elementary matrices.

That is, applying the simplex method to  $M$  corresponds to premultiplication of  $M$  by the product of these elementary matrices.

This product is termed a “check matrix” and will be denoted  $W$ .

Thus there is a square matrix  $W$ , such that  $WM = \hat{M}$  and we want to determine that  $W$ .

For compatibility, the order of  $W$  must be  $(m + 1) \times (m + 1)$  and we note that in the matrix multiplication on the LHS, the first  $m$  entries in a row of the matrix  $W$  multiply the entries from the  $[A_B|A_N|\mathbf{0}|\mathbf{b}]$  part of  $M$ , whereas the last entry in each row of the matrix  $W$  multiplies each of the z-row entries.

## Matrix algebra and LP – 9

So let's partition  $W$  into the form

$$W = \left[ \begin{array}{c|c} W_1 & W_2 \\ \hline W_3 & W_4 \end{array} \right],$$

where  $W_1$  is  $m \times m$ ,  $W_2$  is  $m \times 1$ ,  $W_3$  is  $1 \times m$ , and  $W_4$  is  $1 \times 1$ .

What does our matrix multiplication  $WM = \hat{M}$  now give us?

$$\left[ \begin{array}{c|c} W_1 & W_2 \\ \hline W_3 & W_4 \end{array} \right] \left[ \begin{array}{c|c|c|c} A_B & A_N & \mathbf{0} & \mathbf{b} \\ \hline -\mathbf{c}_B^T & -\mathbf{c}_N^T & 1 & z_0 \end{array} \right] = \left[ \begin{array}{c|c|c|c} I & \hat{A}_N & \mathbf{0} & \hat{\mathbf{b}} \\ \hline \mathbf{0}^T & -\hat{\mathbf{c}}_N^T & 1 & z'_0 \end{array} \right].$$

- From the  $z$ -columns, rows 1 to  $m$ ,  $W_1 \cdot \mathbf{0} + W_2 = \mathbf{0} \implies W_2 = \mathbf{0}$ .
- From  $(m+1)^{st}$  row of the  $z$ -column,  $W_3 \cdot \mathbf{0} + W_4 \cdot 1 = 1 \implies W_4 = 1$ .

## Matrix algebra and LP – 10

So we now have

$$W = \left[ \begin{array}{c|c} W_1 & \mathbf{0} \\ \hline W_3 & 1 \end{array} \right].$$

and so we see that

$$\bullet [W_1 \ \mathbf{0}] \begin{bmatrix} A_B \\ -\mathbf{c}_B^T \end{bmatrix} = I \implies W_1 A_B = I, \text{ and hence } W_1 = A_B^{-1}.$$

$$\bullet [W_3 \ 1] \begin{bmatrix} A_B \\ -\mathbf{c}_B^T \end{bmatrix} = \mathbf{0}^T \implies W_3 A_B - \mathbf{c}_B^T = \mathbf{0}^T \implies W_3 = \mathbf{c}_B^T A_B^{-1}.$$

So we now have

$$W = \left[ \begin{array}{c|c} A_B^{-1} & \mathbf{0} \\ \hline \mathbf{c}_B^T A_B^{-1} & 1 \end{array} \right].$$

## Matrix algebra and LP – 11

Therefore, we write the equation for the final tableau as

$$\begin{aligned}\widehat{M} = WM &= \left[ \begin{array}{c|c} A_B^{-1} & \mathbf{0} \\ \hline \mathbf{c}_B^T A_B^{-1} & 1 \end{array} \right] \left[ \begin{array}{c|c|c|c} A_B & A_N & \mathbf{0} & \mathbf{b} \\ \hline -\mathbf{c}_B^T & -\mathbf{c}_N^T & 1 & z_0 \end{array} \right] \\ &= \left[ \begin{array}{c|c|c|c} I & A_B^{-1} A_N & \mathbf{0} & A_B^{-1} \mathbf{b} \\ \hline \mathbf{0}^T & \mathbf{c}_B^T A_B^{-1} A_N - \mathbf{c}_N^T & 1 & \mathbf{c}_B^T A_B^{-1} \mathbf{b} + z_0 \end{array} \right]\end{aligned}$$

and leaving out the z-column, this becomes

$$= \left[ \begin{array}{c|c|c} I & A_B^{-1} A_N & A_B^{-1} \mathbf{b} \\ \hline \mathbf{0}^T & \mathbf{c}_B^T A_B^{-1} A_N - \mathbf{c}_N^T & \mathbf{c}_B^T A_B^{-1} \mathbf{b} + z_0 \end{array} \right].$$



## Matrix algebra and LP – 12

We now have a way of representing the simplex method as a matrix multiplication.

Given a basis (choice of non-zero variables) we can work out the appropriate tableau by inverting  $A_B$ .

Note that for certain choices of  $B$  we will find  $A_B^{-1}\mathbf{b}$  has negative components and so the basic solution  $\mathbf{x}_B = A_B^{-1}\mathbf{b}$  is not feasible. The simplex algorithm tells us which  $B$ s to consider.

The simplex algorithm employs 'tricks' to compute the inverse of a new  $A_B$  from the old  $A_B^{-1}$  (exploiting the fact that only one variable enters and one leaves).

The matrix format is handy for many applications. In particular, it allows us to consider what happens if the problem is changed slightly, for instance, if some of the constants in  $A$ ,  $\mathbf{b}$  or  $\mathbf{c}$  are changed.

## Matrix algebra and LP – 13

We have

$$\hat{M} = \left[ \begin{array}{c|c|c} I & A_B^{-1}A_N & A_B^{-1}\mathbf{b} \\ \hline \mathbf{0}^T & \mathbf{c}_B^T A_B^{-1}A_N - \mathbf{c}_N^T & \mathbf{c}_B^T A_B^{-1}\mathbf{b} + z_0 \end{array} \right].$$

Recall in our earlier example that we had

$$\hat{M} =$$

$x_1$	$x_2$	$x_4$	$x_3$	$x_5$	$z$	$\mathbf{b}$
1	0	0	$\frac{3}{2}$	$\frac{1}{2}$	0	2
0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	2
0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0
0	0	0	3	2	1	6

and so we can easily write down  $A_B^{-1}A_N$ , and  $A_B^{-1}\mathbf{b}$ .

## Matrix algebra and LP – 14

Recall the dual problem is

$$(D) \quad \min \quad \mathbf{y}^T \mathbf{b} + z_0$$
$$\mathbf{y}^T A \geq \mathbf{c}^T$$

all  $y_i$ , free variables.

Consider letting  $\mathbf{y} = (A_B^{-1})^T \mathbf{c}_B$ , then we have  $A_B^T \mathbf{y} = \mathbf{c}_B$ , and hence if  $A_N^T \mathbf{y} \geq \mathbf{c}_N$ , then  $\mathbf{y}$  is feasible.

Furthermore, consider  $\mathbf{x}_B = A_B^{-1} \mathbf{b}$ ,  $\mathbf{x}_N = 0$ , which is a basic solution for  $(P)$ , and we see that the complementary slackness relation is satisfied since

$$\mathbf{c}_B - A_B^T \mathbf{y} = 0 \implies (\mathbf{c}_B - A_B^T \mathbf{y})^T \mathbf{x}_B = 0$$

$$\mathbf{x}_N = 0 \implies (\mathbf{c}_N - A_N^T \mathbf{y})^T \mathbf{x}_N = 0.$$

## Matrix algebra and LP – 15

Consequently,

$$\mathbf{x}_B = A_B^{-1}b, \mathbf{x}_N = 0 \quad \text{and} \quad \mathbf{y} = (A_B^{-1})^T \mathbf{c}_B$$

are respectively optimal for the primal and dual, provided  $A_N^T \mathbf{y} \geq \mathbf{c}_N$ , which occurs at the optimal solution to (P).

We also have that for these solutions that

$$\mathbf{c}_B^T \mathbf{x}_B = \mathbf{c}_B^T A_B^{-1} \mathbf{b} = \mathbf{y}^T b.$$

We have shown that in general, the objective row of the final (optimal) tableau contains

$$\mathbf{y}^T A_N - \mathbf{c}_N^T,$$

where  $\mathbf{y}$  is the vector of dual variables.

Note also that  $W$  contains the dual variables  $\mathbf{y} = \mathbf{c}_B^T A_B^{-1}$ .

# Takeaways

- The dual problem has an interpretation that is useful
- We can solve problems in the dual domain sometimes more easily
- We can represent LP Tableaus as matrices

# Further reading I