Transform Methods & Signal Processing
Class Exercise 5: solutions

Matthew Roughan
<matthew.roughan@adelaide.edu.au>

1. [10 marks] Figure 1 shows a box diagram of a biquad filter.

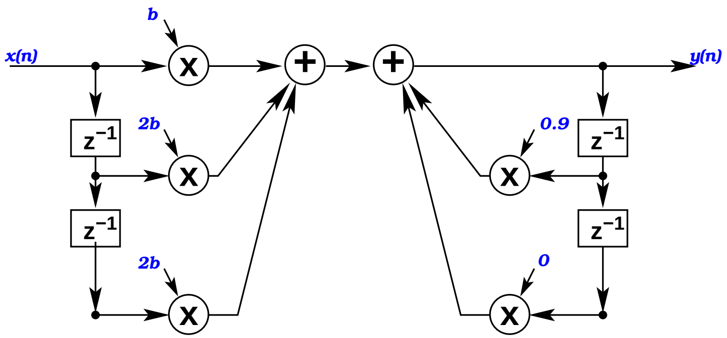


Figure 1: A biquad filter.

(a) Write the z-transform form of the transfer function of the filter.
Solution:

$$H(z) = \frac{B(z)}{A(z)} = b\frac{1+2z^{-1}+2z^{-2}}{1-0.9z^{-1}} = b\frac{z^2+2z^1+2}{z^2-0.9z}$$

See Figure 2 (a) for a picture of the transfer function.

(b) Find the poles and zeros of the filter.
Solution:

$$B(z) = z^{-2}(z-(-1+i))(z-(-1-i))$$

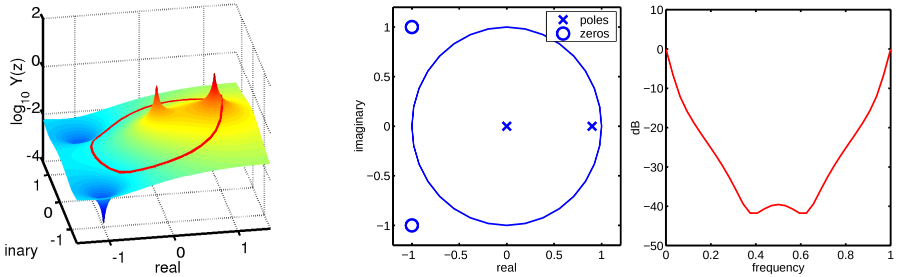
So the zeros are at $(-1 \pm i)$, and

$$A(z) = z^{-2}z(z-0.9)$$

So the poles are at 0 and 0.9.
See Figure 2 (b) for a poles and zeros of the transfer function.

(c) Is the filter stable? Why?
Solution: The poles of the filter are inside the unit circle on the complex plane so the filter is BIBO stable.

(d) Is the filter invertible? Why?
Solution: The zeros of the fitler are outside the unit circle on the complex plane so the filter's inverse is not BIBO stable. *Note this is a correction to the notes, so I dod not deduct marks for incorrect answers to this question.*



(a) z-transform transfer function (b) Poles and zeros (c) Frequency transfer function

Figure 2: The transfer function for the biquad filter.

(e) Is the filter a high-pass, band-pass, or low-pass filter? Why?
Solution: Given the locations of its poles and zeros, it is clearly a low-pass filter. Figure 2 (c) shows the transfer function, where we can see that the filter is a low-pass.

(f) Find (analytically) the impulse response of the filter.
Solution:

$$\begin{aligned} H(z) &= \frac{B(z)}{A(z)} \\ &= b\frac{1+2z^{-1}+2z^{-2}}{1-0.9z^{-1}} \end{aligned}$$

We can expand

$$\frac{1}{1-az^{-1}} = \sum_{k=0}^{\infty} a^k z^{-k}$$

and so the filter can be written

$$\begin{aligned} H(z) &= b\left(1+2z^{-1}+2z^{-2}\right)\sum_{k=0}^{\infty} a^k z^{-k} \\ &= b\left[\sum_{k=0}^{\infty} a^k z^{-k} + 2z^{-1}\sum_{k=0}^{\infty} a^k z^{-k} + 2z^{-2}\sum_{k=0}^{\infty} a^k z^{-k}\right] \\ &= b\left[\sum_{k=0}^{\infty} a^k z^{-k} + 2\sum_{k=0}^{\infty} a^k z^{-k-1} + 2\sum_{k=0}^{\infty} a^k z^{-k-2}\right] \\ &= b\left[\sum_{k=0}^{\infty} a^k z^{-k} + 2\sum_{k=1}^{\infty} a^{k-1} z^{-k} + 2\sum_{k=2}^{\infty} a^{k-2} z^{-k}\right] \\ &= b\left[\sum_{k=0}^{\infty} a^k z^{-k} + 2\sum_{k=0}^{\infty} a^{k-1} z^{-k} + 2\sum_{k=0}^{\infty} a^{k-2} z^{-k} - 2a^{-1} - 2a^{-2} - 2a^{-1}z^{-1}\right] \\ &= b\sum_{k=0}^{\infty}\left[a^k + 2a^{k-1} + 2a^{k-2}\right] z^{-k} - 2b\left[a^{-1} + a^{-2} + a^{-1}z^{-1}\right] \end{aligned}$$

So the impulse response looks like

$$b(1, a+2, a^2+2a+2, a^3+2a^2+2a, \ldots, a^k+2a^{k-1}+2a^{k-2}, \ldots)$$

where $a = 0.9$.

(g) is the filter linear? time-invariant? causal?

Solution: The filter is a linear, time-invariant, causal filter.

(h) What would be the result of passing a signal through two such filters? Draw a box diagram of the new filter (in the same form as that of Figure 1, i.e. not as a cascade of two biquads).

Solution: Placing the two filters in sequence results in a transfer function $H_2(z)$ which is the square of the transfer function of the biquad, e.g.

$$H_2(z) = H^2(z) = b^2 \frac{\left(1 + 2z^{-1} + 2z^{-2}\right)^2}{\left(1 - 0.9z^{-1}\right)^2} = b^2 \frac{1 + 4z^{-1} + 8z^{-2} + 8z^{-3} + 4z^{-4}}{1 - 1.8z^{-1} + 0.81z^{-2}}$$

We can see that this corresponds to a 4th order ARMA filter, which can be drawn in box diagram form as in Figure 3. Note that the original biquad was a low-pass filter, and so this filter must also be a low pass filter (with similar stop band), but that the stop-band attenuation will be the square of that of the biquad.

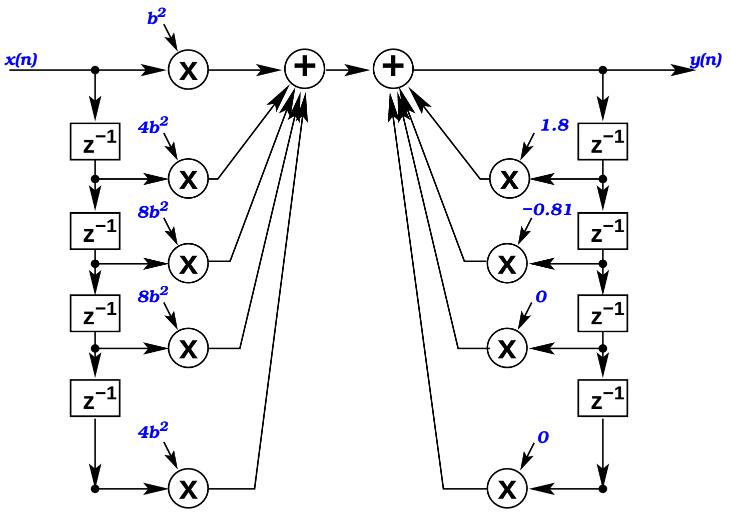


Figure 3: A 4th order filter equivalent to 2 of the biquads defined in Figure 1.

2*. 10 marks A filter is BIBO (Bounded Input, Bounded Output) stable iff the impulse response be absolutely summable, i.e., its L^1 norm exists and is finite, e.g.

$$\sum_{i=-\infty}^{\infty} |w(i)| < \infty.$$

(a) Prove that this is a necessary condition for a filter where $w(i) \geq 0$. Explain how we can generalize to the general case.

Proof: Assume $w(i) \geq 0$, and choose a bounded input signal which is just all ones, i.e. $x(n) = 1$ for all n. Then

$$w * x = \sum_{i=-\infty}^{\infty} w(i) = \sum_{i=-\infty}^{\infty} |w(i)|,$$

which must be finite for the output to be bounded. When $w(i)$ are not all non-negative, then one needs to choose a signal which is $x(i) = \text{sgn}(w(-i))$, and the convolution, at $n = 0$ of the two signals will again be $\sum_{i=-\infty}^{\infty} |w(i)|$.

(b) Prove (using the triangle inequality) that this is a sufficient condition for BIBO.

Proof:

$$\begin{aligned} |y(n)| &= \left| \sum_{k=-\infty}^{\infty} w(k)x(n-k) \right| \\ &\leq \sum_{k=-\infty}^{\infty} |w(k)x(n-k)| \\ &= \sum_{k=-\infty}^{\infty} |w(k)|\,|x(n-k)| \end{aligned}$$

by the triangle inequality. Take $\|x\|_\infty = \max |x(n)|$ then

$$\begin{aligned} |y(n)| &\leq \sum_{k=-\infty}^{\infty} |w(k)|\, \|x\|_\infty \\ &\leq \|x\|_\infty \sum_{k=-\infty}^{\infty} |w(k)| \\ &< \infty \end{aligned}$$

when the condition holds and the input is bounded.

(c) It is often convenient to characterize BIBO filters in the frequency domain (i.e. using a z-transform), i.e., by noting that all of the filter's poles must be inside the unit circle (in the complex plane). The proof revolves around showing that the z-transform of a BIBO filter must converge on the unit circle – show this is the case.

Proof: Take the z-transform on the unit circle in the complex plane, i.e. $z = e^{i\theta}$, so

$$\begin{aligned} W(z) &= \sum_{k=-\infty}^{\infty} w(k)z^{-k} \\ &= \sum_{k=-\infty}^{\infty} w(k)e^{-ik\theta} \end{aligned}$$

Now (again using the triangle inequality)

$$\begin{aligned} |W(z)| &= \left| \sum_{k=-\infty}^{\infty} w(k)e^{-ik\theta} \right| \\ &= \sum_{k=-\infty}^{\infty} |w(k)|\,|e^{-ik\theta}| \\ &= \sum_{k=-\infty}^{\infty} |w(k)| \\ &< \infty \end{aligned}$$