Transform Methods & Signal Processing

lecture 11

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2D Wavelets

Generalizing Wavelets to 2D is not quite as simple as generalizing the Fourier transform to higher dimensions.

We extend Wavelets to 2D

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2D

Simple extension (for separable wavelet bases)

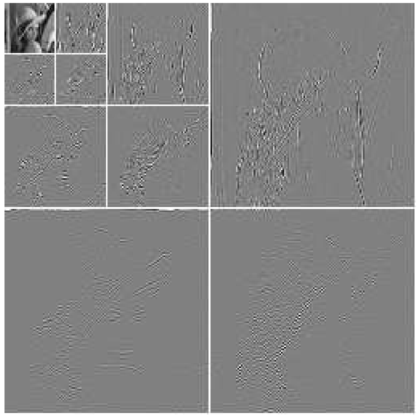

Figure 2.23 from Mallat (p.310), made using WaveLab http://www-stat.stanford.edu/~wavelab/

```
% file:       lena.m, (c) Matthew Roughan, Mon Aug 14 2006
%
Image = ReadImage('Lenna');
figure(1)
imagesc(Image);
colormap(gray);
opts = struct('height',8, 'Color', 'gray');
axis image
axis('off')
set(gca,'DataAspectRatio', [1 1 1]);
exportfig(gcf,sprintf('Plots/lena.eps'), opts, 'format', 'eps');

[n,J] = quadlength(Image);
qmf = MakeONFilter('Daubechies',8);
L = 5;
wc = FWT2_PO(Image,L,qmf);

figure(3)
Display2dProjV(wc,L,qmf);
axis image
axis('off')
set(gca,'DataAspectRatio', [1 1 1]);
exportfig(gcf,sprintf('Plots/lena_approx.eps'), opts, 'format', 'eps');

% taken from wt07fig26 (Mallat Fig 7.26)
wc2 = wc;
avg = wc(1:2^L,1:2^L);
wc2(1:2^L,1:2^L) = 1-(avg./max(max(abs(avg))));
wc2(1:2^L,2^L) = zeros(2^L,1);
wc2(2^L,1:2^L) = zeros(1,2^L);
```

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Separable wavelet bases

Take any orthonormal wavelet basis $\{\psi_{n,j}\}_{n,j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$, then a separable wavelet basis for $L^2(\mathbb{R}^2)$ is

$$\{\psi_{n_1,j_1}\psi_{n_2,j_2}\}_{n_1,n_2,j_1,j_2\in\mathbb{Z}}$$

- but basis above mixes resolutions at different scales j_1 and j_2
- separable MRAs lead to constructions that are products of functions dilated to the same scale
- can construct non-separable bases, but used less often
- build approximation spaces $V_j^2 = V_j \otimes V_j$ such that these are separable, i.e., basis looks like $\phi^2_{n_1,n_2,j}(x_1,x_2) = \phi_{n_1,j}(x_1)\phi_{n_2,j}(x_2)$

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2D scaling functions

Scaling functions

$$\phi^2_{n_1,n_2,j}(x_1,x_2) = \phi_{n_1,j}(x_1)\phi_{n_2,j}(x_2) = \frac{1}{2^j}\phi\left(\frac{x_1}{2^j} - n_1\right)\phi\left(\frac{x_2}{2^j} - n_2\right)$$

Approximation $\hat{f}_j = \sum_{n_1,n_2,j} \left\langle f, \phi^2_{n_1,n_2,j} \right\rangle \phi^2_{n_1,n_2,j}$ where

$$\left\langle f, \phi^2_{n_1,n_2,j} \right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1,x_2)\phi^2_{n_1,n_2,j}(x_1,x_2)\, dx_1\, dx_2$$

which separates into two integrals if f separates.

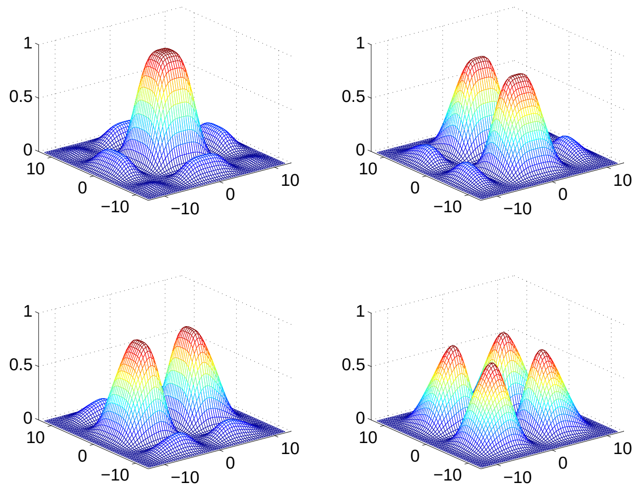
2D Wavelets

We get 3 mother wavelets

$$\begin{aligned} \psi^1(x_1,x_2) &= \phi(x_1)\psi(x_2) \\ \psi^2(x_1,x_2) &= \psi(x_1)\phi(x_2) \\ \psi^3(x_1,x_2) &= \psi(x_1)\psi(x_2) \end{aligned}$$

from which we derive the wavelets by dilation and 2D translations.

2D Wavelet filter spectrum



2D MRA tree

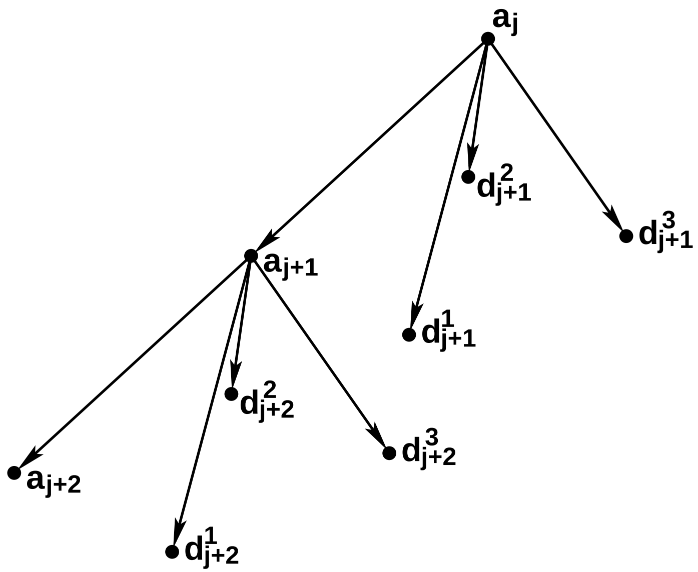


Figure 2.24 from Mallat (p.308), made using WaveLab `http://www-stat.stanford.edu/~wavelab/`

2D Wavelet Filters

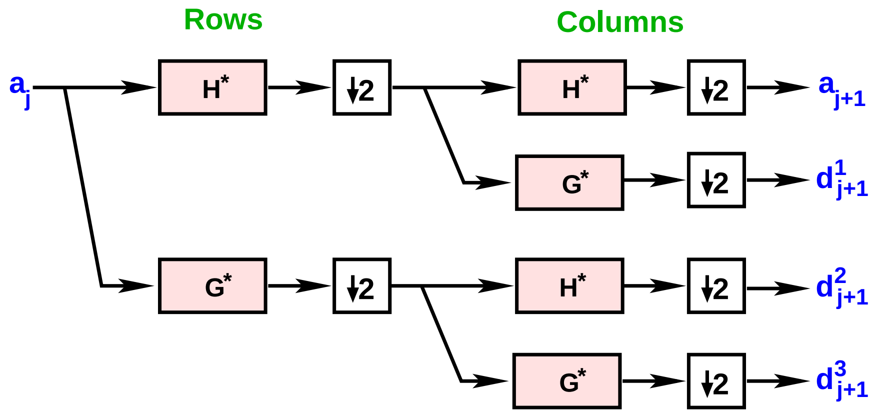
$$a_j(n_1,n_2) = \langle f, \phi_{n1,n2,j} \rangle \quad \text{and} \quad d_j^k(n_1,n_2) = \langle f, \psi^k_{n1,n2,j} \rangle$$

one step of the decomposition takes the form

$$a_{j+1}(n_1,n_2) = \left[a_j * \overline{hh}\right](2n_1,2n_2)$$
$$d^1_{j+1}(n_1,n_2) = \left[a_j * \overline{h}\overline{g}\right](2n_1,2n_2)$$
$$d^2_{j+1}(n_1,n_2) = \left[a_j * \overline{g}\overline{h}\right](2n_1,2n_2)$$
$$d^3_{j+1}(n_1,n_2) = \left[a_j * \overline{gg}\right](2n_1,2n_2)$$

- Notation $\overline{h}(n) = h(-n)$
- Product hg means $[hg](n_1,n_2) = h(n_1)g(n_2)$
- 2D convolution can be performed as two 1D convolutions (downsample between doing rows and columns)

2D Wavelet Block Diagram



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2D Wavelet Reconstruction

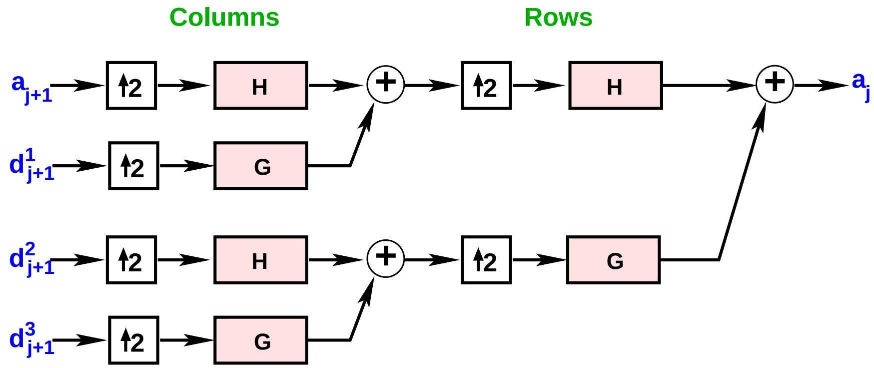
Representation $\{a_J, \{d_j^1, d_j^2, d_j^3\}_{L<j\leq j}\}$

- define upsampled 2D image $\breve{y}(n_1, n_2)$ made by inserting rows and columns of zeros in between existing rows and columns

Reconstruction

$$a_j = \breve{a}_{j+1} * hh + \breve{d}^1{}_{j+1} * hg + \breve{d}^2{}_{j+1} * gh + \breve{d}^3{}_{j+1} * gg$$

2D Wavelet Block Diagram



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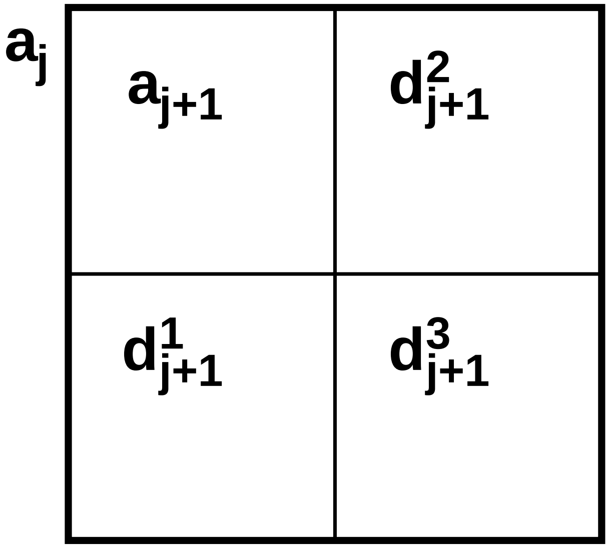
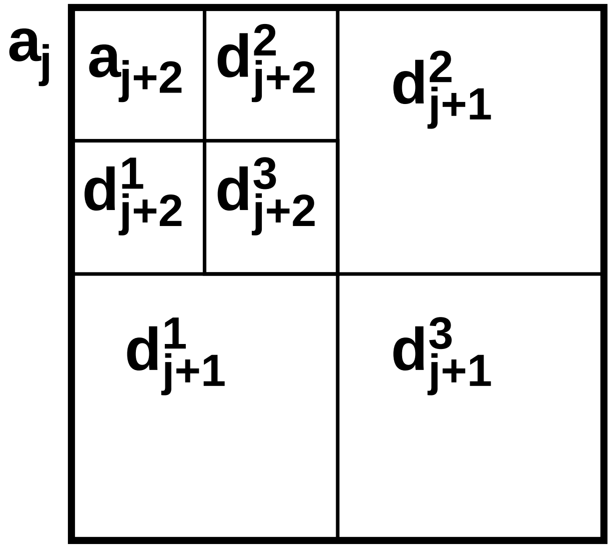
2D Layout



2D Layout



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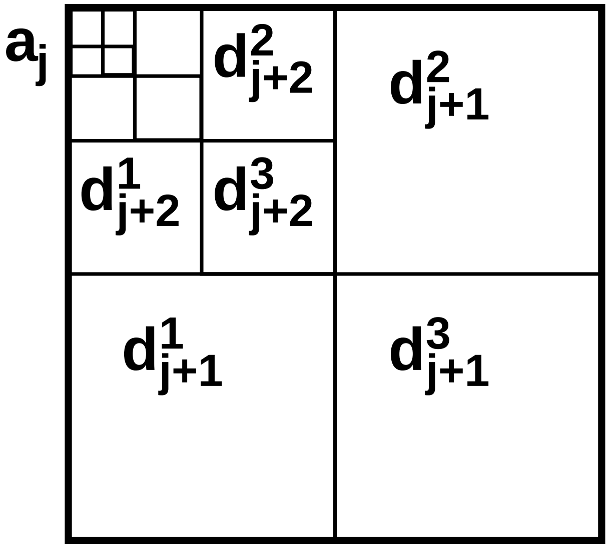
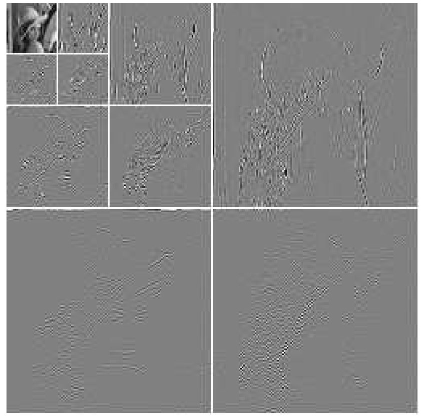
2D Layout



2D

Simple extension (for separable wavelet bases)



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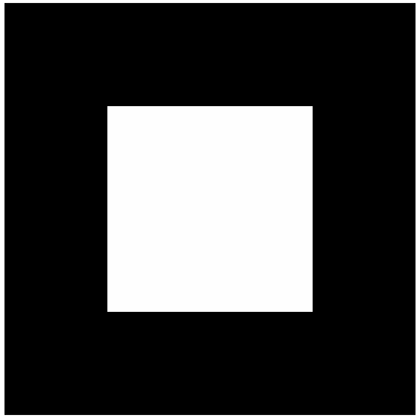

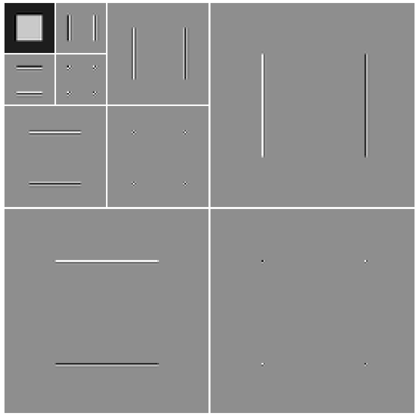

Applications

Figure 2.23 from Mallat (p.310), made using WaveLab `http://www-stat.stanford.edu/~wavelab/`

```
% file:       wavelet_box.m, (c) Matthew Roughan, Mon Aug 14 2006
%
N = 256;
Image = MakeImage('Square',N);
figure(1)
imagesc(Image);
colormap(gray);
opts = struct('height',8, 'Color', 'gray');
axis image
axis('off')
set(gca,'DataAspectRatio', [1 1 1]);
exportfig(gcf,sprintf('Plots/wavelet_box.eps'), opts, 'format', 'eps');

[n,J] = quadlength(Image);
qmf = MakeONFilter('Daubechies',8);
L = 5;
wc = FWT2_PO(Image,L,qmf);

figure(3)
Display2dProjV(wc,L,qmf);
axis image
axis('off')
set(gca,'DataAspectRatio', [1 1 1]);
exportfig(gcf,sprintf('Plots/wavelet_box_approx.eps'), opts, 'format', 'eps');

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wc2 = wc;
avg = wc(1:2^L,1:2^L);
wc2(1:2^L,1:2^L) = 1-(avg./max(max(abs(avg))));
wc2(1:2^L,2^L) = zeros(2^L,1);
```

Compression: FBI fingerprints

- FBI have $\sim$ 30 million fingerprints
  - actually more like 200 million (repeats etc)
  - 30-50,000 more per day
- 1993 started converting from ink on cards (transmitted by fax) to digital storage
- 500 pixels per inch resolution. 256 grey levels (8 bits)
  - one fingerprint, 700,000 pixels, and 6 MB storage
  - 200 TB for whole database
  - have to transmit cards (3 hours on slow modem)
- compression is needed

Compression: FBI fingerprints

Basic idea, quantize in the transform space.

- use Wavelet transform (in 2D)
- steps
  - wavelet transform
  - quantize coefficients
  - entropy encoding
- called WSQ (Wavelet Scalar Quantization)

Links:
http://www.c3.lanl.gov/~brislawn/FBI/FBI.html
http://www.amara.com/IEEEwave/IW_fbi.html
ftp://ftp.c3.lanl.gov/pub/misc/WSQ/FBI_WSQ_FAQ

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Compression: FBI fingerprints

Compression: FBI fingerprints

Original, file size 589,824 bytes.

JPEG, file size 45853 bytes, compression ratio 12.9.

http://www.c3.lanl.gov/~brislawn/FBI/FBI.html

Compression: FBI fingerprints

Original, file size 589,824 bytes.

Wavelets, compression ratio 12.9.

http://www.c3.lanl.gov/~brislawn/FBI/FBI.html

Compression

Compression

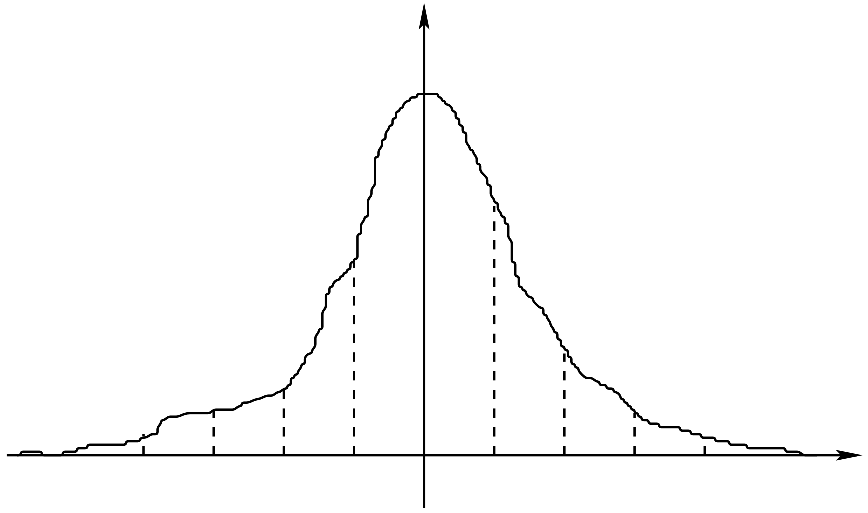
- Simplest version – just perform algorithm above
- more general: quantize wavelet coefficients by a fixed step

$$\hat{d}(j,k) = Q\text{sign}(d(j,k)) \left\lfloor \frac{|d(j,k)|}{Q} \right\rfloor$$

- more general: use a quantization table
- Inverse Wavelet Transform of $\{\hat{d}(j,k)\}_{j,k}$

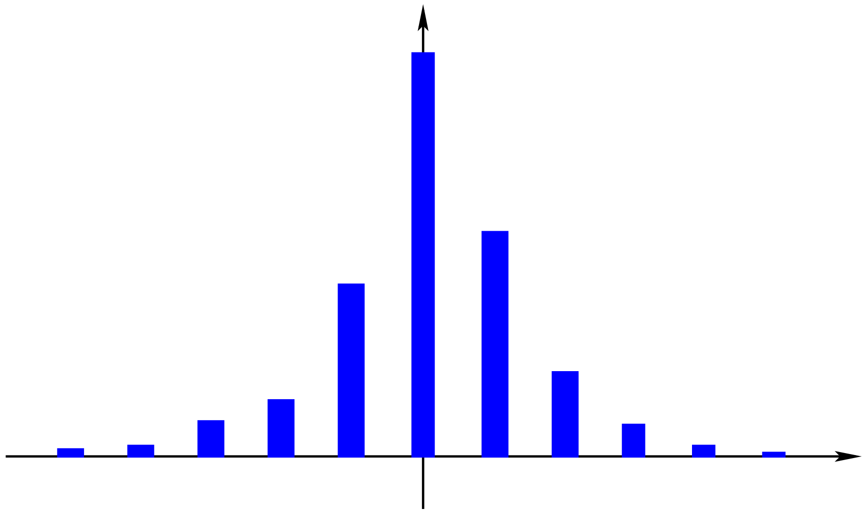
Quantization

We start with some continuous distribution



Quantization

Then quantize the distribution into a number of levels



JPEG 2000

JPEG 2000 uses wavelets rather than the DCT

Links:
http://www.jpeg.org/jpeg2000/
http://www.gvsu.edu/math/wavelets/student_work/Miljour/

Some comparisons
http://www.imagepower.com/technology/jpeg2000/compare/
http://www.levien.com/gimp/jpeg2000/comparison.html
http://ai.fri.uni-lj.si/~aleks/jpeg/artifacts.htm
http://www.fnordware.com/j2k/jp2samples.html