Variational Methods and Optimal Control
Class Exercise 2 solutions

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1. For the fixed end point problem to find the extremals of a functional

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y')\, dx$$

where f has continuous partial derivatives of up to second order wrt x, y and y', state the Euler-Lagrange equation that the extremal curve must statisfy.

Solution: The Euler-Lagrange equation is $\boxed{\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} \;=\; 0}$

2. Find the extremals of the functionals

(a) $F\{y\} = \int_a^b \frac{\sqrt{1+y'^2}}{y}\, dx$

Solution:

Note that $f = f(y, y') = \frac{\sqrt{1+y'^2}}{y}$ is independent of x, and so we can find constant function

$$H(y, y') = y'\frac{\partial f}{\partial y'} - f(y, y') = \frac{y'^2}{y\sqrt{1+y'^2}} - \frac{\sqrt{1+y'^2}}{y} = c_1$$

Multiplying both sides by $y\sqrt{1+y'^2}$ we get

$$y'^2 - (1 + y'^2) = 1 = c_1 y\sqrt{1+y'^2}$$

and squaring we get

$$1 = c_1^2 y^2 (1 + y'^2)$$

This can be rearranged to the form

$$y'^2 = \frac{1 - c_1^2 y^2}{c_1^2 y^2}$$

or

$$dx = \pm \frac{c_1 y}{\sqrt{1 - c_1^2 y^2}}\, dy$$

Integrating we get

$$\begin{aligned} x &= \int \frac{c_1 y}{\sqrt{1-c_1^2 y^2}}\, dy \\ &= \pm\frac{\sqrt{1-c_1^2y^2}}{c_1} + c_2 \end{aligned}$$

Hence,

$$(x - c_2)^2 + y^2 = \frac{1}{c_1^2}$$

So the extremals are circles (with center along the x-axis).

(b) $F\{y\} = \int_0^1 \left[xy^2 + (y + x^2 y)y'\right]\, dx$, subject to $y(0) = 0$, and $y(1) = 2$.

Solution:

The function $f = f(x, y, y') = A(x, y)y' + B(x, y)$, where

$$\begin{aligned} A(x,y) &= y + x^2 y \\ B(x,y) &= xy^2 \end{aligned}$$

and so f is linear in y', so this is the degenerate case, and so the E-L equations reduce to

$$\frac{\partial A}{\partial x} - \frac{\partial B}{\partial y} \;=\; 0$$

Now

$$\begin{aligned} \frac{\partial A}{\partial x} &= 2xy \\ \frac{\partial B}{\partial y} &= 2xy \end{aligned}$$

so the E-L equations reduce to

$$\frac{\partial A}{\partial x} - \frac{\partial B}{\partial y} = 2xy - 2xy = 0$$

which is an identity (it is always true), and so $F\{y\}$ does not depend on the path (curve), and so all curves that join the end-points satisfy the E-L equations.

(c) $F\{y\} = \int_0^1 \left[xy^2 + (y + xy^2)y'\right]\, dx$, subject to $y(0) = 0$, and $y(1) = 2$.

Solution:

The function $f = f(x, y, y') = A(x, y)y' + B(x, y)$, where

$$\begin{aligned} A(x,y) &= y + xy^2 \\ B(x,y) &= xy^2 \end{aligned}$$

and so f is linear in y', so this is the degenerate case, and so the E-L equations reduce to

$$\frac{\partial A}{\partial x} - \frac{\partial B}{\partial y} \;=\; 0$$

Now

$$\begin{aligned} \frac{\partial A}{\partial x} &= y^2 \\ \frac{\partial B}{\partial y} &= 2xy \end{aligned}$$

so the E-L equations reduce to

$$\frac{\partial A}{\partial x} - \frac{\partial B}{\partial y} = y^2 - 2xy = y(1 - 2x) = 0$$

Thus the only possible extremals are $y = 0$ and $y = 2x$. The variation problem has an extremal only if one of these curves passes through the required end-points. Fortunately, $y = 2x$ passes through the two end points, and so the straight line $y = 2x$ is the required extremal.

3. A functional F is given by

$$F[y] \;=\; \int_0^1 x(1-x)yy''\,dx.$$

Use an appropriate integration by parts to show that F can be expressed in the standard form

$$F[y] \;=\; \int_0^1 f(x,y,y')\,dx$$

and derive an ordinary differential equation that must be satisfied by any extremal to F for fixed values of $y(0)$ and $y(1)$.

Solution: Using integration by parts provides

$$\begin{aligned} F[y] &= \int_0^1 x(1-x)y.y'' \\ &= [x(1-x)y.y']_0^1 - \int_0^1 [(1-2x)y + x(1-x)y']\,y'\,dx \\ &= 0 + \int_0^1 [(2x-1)y + x(x-1)y']\,y'\,dx, \end{aligned}$$

which is in the standard form $\int_a^b f(x,y,y')\,dx$. The E–L equation

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

reads

$$(2x-1)y' = \frac{d}{dx}\,[(2x-1)y + 2x(x-1)y']$$

or

$$0 = 2y + (4x-2)y' + 2x(x-1)y'',$$

that is,

$$x(1-x)y'' + (1-2x)y' - y = 0.$$

4. State if the following functionals are autonomous, degenerate, and/or have dependence on y.

(a) $F\{y\} = \int_a^b \frac{\sqrt{1+y'^2}}{y}\,dx$

(b) $F\{y\} = \int_a^b y^2y' + xy'\,dx$

(c) $F\{y\} = \int_a^b \cos(xy') + \sin(xy')\,dx$

(d) $F\{y\} = \int_a^b \cos^2(y') + \sin^2(xy)\,dx$

Solution: Note that the last problem is a trick: $\sin^2 + \cos^2 = 1$ regardless of the arguments to the functions, so this integral is constant.

problem	autonomous	degenerate	dependent on y
$F\{y\} = \int_a^b \frac{\sqrt{1+y'^2}}{y}\,dx$	yes	no	yes
$F\{y\} = \int_a^b y^2y' + xy'\,dx$	no	yes	yes
$F\{y\} = \int_a^b \cos(xy') + \sin(xy')\,dx$	no	no	no
$F\{y\} = \int_a^b \cos^2(y') + \sin^2(xy)\,dx$	yes	yes	no

5. Find the shape of a geodesic on the (curved part) of the surface of a cylinder.

Can you explain the geodesic by "unrolling" the cylinder?

Solution:

From lectures, the general geodesic formulation

$$\begin{aligned} L &= \int \sqrt{P + 2Qv' + Rv'^2}\,du \\ &= \int \sqrt{Pu'^2 + 2Qu' + R}\,dv \end{aligned}$$

where $u' = du/dv$ and $v' = dv/du$ and

$$\begin{aligned} P &= \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 \\ Q &= \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} \\ R &= \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 \end{aligned}$$

The Euler-Lagrange equations are

$$\frac{\frac{\partial P}{\partial v} + 2v'\frac{\partial Q}{\partial v} + v'^2\frac{\partial R}{\partial v}}{2\sqrt{P + 2Qv' + Rv'^2}} - \frac{d}{du}\left(\frac{Q + Rv'}{\sqrt{P + 2Qv' + Rv'^2}}\right) = 0$$

Standard co-ordinates on the cylinder (radius 1) are

$$\begin{aligned} x &= u \\ y &= \cos(\theta) \\ z &= \sin(\theta) \end{aligned}$$

$$\begin{aligned} x_u &= 1 & x_\theta &= 0 \\ y_u &= 0 & y_\theta &= -\sin(\theta) \\ z_u &= 0 & z_\theta &= \cos(\theta) \end{aligned}$$

$$P = 1 \qquad Q = 0 \qquad R = (-\sin(\theta))^2 + \cos(\theta)^2 = 1$$

E-L equation

$$\frac{\theta'}{\sqrt{1+\theta'}} = const$$

which implies that

$$\theta' = c_1$$

And hence

$$\theta(u) = c_1 u + c_2$$

The angle is proportional to the distance around the cylinder.

These describe spirals around the curved part of the cylinder.

We can also note that if we unrolled the cylinder then (u,θ) would be a point on the 2D Euclidean plane so that the curve

$$\theta(u) = c_1 u + c_2$$

is just a straight line.