Variational Methods and Optimal Control
Class Exercise 5 solutions

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1. Find the coordinates of the point(s) nearest the origin on the surface $xyz = a^3$, for $x, y, z \geq 0$.

Show (using the transversal conditions and the Euler-Lagrange equations) that if we were to draw a line between this point and the origin, it would be a transversal of minimum length between the origin and the surface.

Solution: We know from previous work that shortest-paths in Euclidean space are straight lines, which arise from minimizing the functional

$$F\{x, y, z\} = \int_0^S ds = \int_0^T \sqrt{x'^2 + y'^2 + z'^2}\, dt.$$

Using the free end-point conditions for several dependent variables we get

$$\sum_{k=1}^n p_k \delta q_k - H\delta t = 0 \text{ where } p_k = \frac{\partial L}{\partial \dot{q}_k} \text{ and } H = \sum_{k=1}^n \dot{q}_k p_k - L$$

From this we can define

$$\begin{aligned}
p_x &= \frac{\partial L}{\partial x'} = \frac{x'}{\sqrt{x'^2 + y'^2 + z'^2}} \\
p_y &= \frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{x'^2 + y'^2 + z'^2}} \\
p_z &= \frac{\partial L}{\partial z'} = \frac{z'}{\sqrt{x'^2 + y'^2 + z'^2}} \\
H &= x'\frac{\partial L}{\partial x'} + y'\frac{\partial L}{\partial y'} + z'\frac{\partial L}{\partial z'} - L = \frac{x'^2 + y'^2 + z'^2}{\sqrt{x'^2 + y'^2 + z'^2}} - \sqrt{x'^2 + y'^2 + z'^2} \\
&= 0
\end{aligned}$$

so the end-point condition is

$$p_x \delta x + p_y \delta y + p_z \delta z = 0.$$

Specify the surface Γ parametrically by $(x_\Gamma(u, v), y_\Gamma(u, v), z_\Gamma(u, v)))$, then we can write

$$\begin{aligned}
\delta x &= \delta u \frac{\partial x_\Gamma}{\partial u} + \delta v \frac{\partial x_\Gamma}{\partial v} \\
\delta y &= \delta u \frac{\partial y_\Gamma}{\partial u} + \delta v \frac{\partial y_\Gamma}{\partial v} \\
\delta z &= \delta u \frac{\partial z_\Gamma}{\partial u} + \delta v \frac{\partial z_\Gamma}{\partial v}
\end{aligned}$$

and so $p_x \delta x + p_y \delta y + p_z \delta z$ is

$$\frac{\delta u}{\sqrt{x'^2 + y'^2 + z'^2}} \left[\frac{dx}{ds}\frac{\partial x_\Gamma}{\partial u} + \frac{dy}{ds}\frac{\partial y_\Gamma}{\partial u} + \frac{dz}{ds}\frac{\partial z_\Gamma}{\partial u} \right] + \frac{\delta v}{\sqrt{x'^2 + y'^2 + z'^2}} \left[\frac{dx}{ds}\frac{\partial x_\Gamma}{\partial v} + \frac{dy}{ds}\frac{\partial y_\Gamma}{\partial v} + \frac{dz}{ds}\frac{\partial z_\Gamma}{\partial v} \right].$$

We can vary δu and δv independently, and the above terms must all be zero, so we get two conditions

$$\begin{aligned}
\frac{d}{ds}(x, y, x) \cdot \mathbf{r}_u &= \frac{dx}{ds}\frac{\partial x_\Gamma}{\partial u} + \frac{dy}{ds}\frac{\partial y_\Gamma}{\partial u} + \frac{dz}{ds}\frac{\partial z_\Gamma}{\partial u} \\
&= 0 \\
\frac{d}{ds}(x, y, x) \cdot \mathbf{r}_v &= \frac{dx}{ds}\frac{\partial x_\Gamma}{\partial v} + \frac{dy}{ds}\frac{\partial y_\Gamma}{\partial v} + \frac{dz}{ds}\frac{\partial z_\Gamma}{\partial v} \\
&= 0,
\end{aligned}$$

where $\mathbf{r}_u$ and $\mathbf{r}_v$ of the two tangential vectors to the surface.

$$\begin{aligned}
\mathbf{r}_u &= \left(\frac{\partial x_\Gamma}{\partial u}, \frac{\partial y_\Gamma}{\partial u}, \frac{\partial z_\Gamma}{\partial u} \right) \\
\mathbf{r}_v &= \left(\frac{\partial x_\Gamma}{\partial v}, \frac{\partial y_\Gamma}{\partial v}, \frac{\partial z_\Gamma}{\partial v} \right)
\end{aligned}$$

The condition above, simply stated says that the extremal curve will join the surface in such a way that its dot product with the tangential vectors is zero, and hence the extremal will be normal to the surface at the point of contact.

Now recall that the normal to a parameterized surface $(x_\Gamma(u, v), y_\Gamma(u, v), z_\Gamma(u, v)))$ can be found by taking the cross products $\mathbf{r}_u \times \mathbf{r}_v$ of the two tangential vectors (simply this can be understood by noting that the cross-product will be at right angles to both tangential vectors):

Or we can derive the normal by noting that the surface can be expressed as a constrait $g(z, y, z) = xyz - a^3 = 0$, and that the normal direction can be derived by grad g, i.e.,

$$\nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right) = (yz, xz, xy).$$

Noting the condition that the extremal curve must join the surface as a normal, and that the extremal curve will be a straight line through the origin, i.e., it will have the parametric form $(\alpha s, \beta s, \gamma s)$, we see that the point of contact with the surface will have normal

$$(yz, xz, xy) = (\beta\gamma s^2, \alpha\gamma s^2, \alpha\beta s^2),$$

and the only way the line's (parameterized) slope (α, β, γ) can match this is if $\alpha = \beta = \gamma$, i.e., we meet the surface at the point (a, a, a).

Of course there is a much simpler solution to this problem. Consider a set of expanding spheres. Obviously, the radii of these spheres form geodesics, and equally clearly the sphere will touch the surface at a tangent, and hence the extremal curve of interest is the radii of the sphere to the point where it just touches the surface (which will be a normal to the surface). This corresponds to minimizing $x^2 + y^2 + z^2$ subject to $xyz = a^3$ and we know the solution to this.

2. **Optimal Control:** 5 marks Minimize

$$F\{u\} = \int_0^1 u^2\, dt$$

subject to

$$\begin{aligned}
\dot{x_1} &= u - x_2 \\
\dot{x_2} &= -u
\end{aligned}$$

and

$$\begin{aligned}
x_1(0) &= 2 \\
x_1(1) &= 1 \\
x_2(0) &= 0 \\
x_2(1) &= 1
\end{aligned}$$

Solution: Including the constraints via Lagrange mulitpliers p_1 and p_2 we get seek to optimize

$$J\{u, x_1, x_2\} = \int_0^1 u^2 + p_1(\dot{x_1} - u + x_2) + p_2(\dot{x_2} + u)\, dt.$$

The Euler-Lagrange equations are

$$\begin{aligned} 2u - p_1 + p_2 &= 0 \\ \dot{p}_1 &= 0 \\ \dot{p}_2 - p_1 &= 0. \end{aligned}$$

Solving the second two we get

$$\begin{aligned} p_1 &= c_1 \\ p_2 &= c_1 t + c_2. \end{aligned}$$

Substituting p_1 and p_2 from above into the first E-L equation $2u = p_1 - p_2$ gives

$$u = \frac{p_1 - p_2}{2} = -\frac{1}{2}c_1 t + \frac{1}{2}(c_1 - c_2).$$

We can substitute this into the system DEs and we get

$$\begin{aligned} \dot{x_1} &= -\frac{1}{2}c_1 t + \frac{1}{2}(c_1 - c_2) - x_2 \\ \dot{x_2} &= \frac{1}{2}c_1 t - \frac{1}{2}(c_1 - c_2), \end{aligned}$$

which we solve to get

$$\begin{aligned} x_2 &= \frac{1}{4}c_1 t^2 - \frac{1}{2}(c_1 - c_2)t + c_3 \\ \dot{x_1} &= -\frac{1}{2}c_1 t + \frac{1}{2}(c_1 - c_2) - \frac{1}{4}c_1 t^2 + \frac{1}{2}(c_1 - c_2)t - c_3 \\ &= \frac{1}{2}(c_1 - c_2) - \frac{1}{4}c_1 t^2 - \frac{1}{2}c_2 t - c_3 \\ x_1 &= c_4 + \left[\frac{1}{2}(c_1 - c_2) - c_3\right] t - \frac{1}{12}c_1 t^3 - \frac{1}{4}c_2 t^2. \end{aligned}$$

Substitute the end-point conditions $x_1(0) = 2$, $x_1(1) = 1$, $x_2(0) = 0$ and $x_2(1) = 1$ and we get four equations

$$\begin{aligned} c_4 &= 2 \\ c_4 + \left[\frac{1}{2}(c_1 - c_2) - c_3\right] - \frac{1}{12}c_1 - \frac{1}{4}c_2 &= 1 \\ c_3 &= 0 \\ -\frac{1}{4}c_1 + \frac{1}{2}c_2 + c_3 &= 1 \end{aligned}$$

From which we get $c_3 = 0$ and $c_4 = 2$, and two remaining equations

$$\begin{aligned} \frac{5}{12}c_1 - \frac{3}{4}c_2 &= -1 \\ -\frac{1}{4}c_1 + \frac{1}{2}c_2 &= 1 \end{aligned}$$

Solving we get $c_1 = 12$ and $c_2 = 8$, which determines u, x_1 and x_2, e.g.,

$$u = -\frac{1}{2}c_1 t + \frac{1}{2}(c_1 - c_2) = -6t + 2.$$

Hence the integral

$$F\{u\} = \int_0^1 u^2\, dt = \int_0^1 (-6t + 2)^2, dt = \int_0^1 36t^2 - 24t + 4, dt = \left[12t^3 - 12t^2 + 4t\right]_0^1 = 4$$

3. Optimal Control: 5 marks Find the minimum value of

$$F\{u\} = x(1) + \int_0^1 \alpha u^2\, dt,$$

where $\alpha > 0$, $x(0) = 0$, $x(1)$ free, and

$$\dot{x} = u.$$

How does the answer change if we add the condition that $|u(t)| \leq 1$?

Solution: Augment the functional with a Lagrange multiplier to get

$$J\{u\} = x(1) + \int_0^1 \alpha u^2 + p(\dot{x} - u)\, dt.$$

The Euler-Lagrange equations will be

$$\begin{aligned} \dot{p} &= 0 \\ 2\alpha u - p &= 0. \end{aligned}$$

Clearly $p = const$, and hence $u = p/2\alpha$ also constant. We can substitute this into the system DE $\dot{x} = u$ to get

$$x = \frac{p}{2\alpha}t + k.$$

Using the initial condition we get $k = 0$. The natural boundary condition for the free end point at t_1 is

$$\sum_i \left(\frac{\partial \phi}{\partial x_i} + \frac{\partial f}{\partial \dot{x}_i}\right)\delta x_i\Big|_{t=t_1} + \left(\frac{\partial \phi}{\partial t} - H\right)\delta t\Big|_{t=t_1} = 0$$

where the final time is fixed as $t_1 = 1$ so $\delta t = 0$ the terminal cost $\phi(x) = x$, and $\partial f/\partial \dot{x} = p$ so the condition is

$$\left(\frac{\partial \phi}{\partial x} + p\right)\Big|_{t=1} = (1 + p)|_{t=1} = 0,$$

but p is constant, so $p = -1$ for all t, and hence

$$x = \frac{-1}{2\alpha}t,$$

and

$$u = \frac{-1}{2\alpha},$$

and the minimal value of the integral

$$F\{u\} = x(1) + \int_0^1 \alpha u^2\, dt = \frac{-1}{2\alpha} + \int_0^1 \frac{1}{4\alpha}\, dt = \frac{-1}{2\alpha} + \frac{1}{4\alpha} = \frac{-1}{4\alpha}.$$

If $|u(t)| \leq 1$ then there are two possibilities.

(a) If $\frac{1}{2\alpha} \leq 1$ then the above solution holds.

(b) If $\frac{1}{2\alpha} > 1$ then the control cannot be $u = -1/2\alpha$, and so will sit on the boundary, i.e., $u = -1$. so this will give the solution $x = -t$, and

$$F\{u\} = -1 + \int_0^1 \alpha\, dt = -1 + \alpha$$