ADELAIDE UNIVERSITY AUSTRALIA

Examination in School of Mathematical Sciences

Semester 2, 2010

006128	Variational Methods and Optimal Control
	APP MATH 3010

Official Reading Time: 10 mins
Writing Time: 180 mins
Total Duration: 190 mins

NUMBER OF QUESTIONS: 6 TOTAL MARKS: 60

Instructions

- Answer ALL questions.
- Begin each answer on a new page.
- Examination materials must not be removed from the examination room.

Materials

- 1 Blue books are provided.
- Calculators are NOT permitted.
- 2 double sided pages of handwritten notes are allowed.

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO.

1. State if the following functionals are, or are not autonomous, degenerate, and/or have explicit dependence on y.

   (a) $F\{y\} = \displaystyle\int_a^b \frac{\sqrt{1+y'^2}}{x}\,dx.$

   (b) $F\{y\} = \displaystyle\int_a^b \sin(y)y' + xy'\,dx.$

   (c) $F\{y\} = \displaystyle\int_a^b \sin(xy')\,dx.$

   (d) $F\{y\} = \displaystyle\int_a^b yy'(1+y')\,dx.$

   Please provide your answer in the form of a table whose rows correspond to each functional, and which has a column for each case. Fill in all parts of the table.

   [12 marks]

2. Find a smooth extremal of

$$F\{y\} = \int_0^4 x y'^3 \, dx$$

such that $y(0) = 0$ and $y(4) = 2$.

[8 marks]

Please turn over for page 4

3. (a) Given the problem to find an extremal curve (x, y)

$$F\{x, y\} = \int f(t, x, y, \dot{x}, \dot{y})\, dt.$$

write the general form of the Euler-Lagrange equations.

(b) Use the Euler-Lagrange equations to derive the shape of the extremal curves for the following functional

$$F\{x, y\} = \int \sqrt{\dot{x}^2 + \dot{y}^2}\, dt.$$

(c) Using the above, and transversailty conditions find the shortest path from the origin to the line $y = -x/2 + 6$, and illustrate the result with a graph.

[10 marks]

4. Consider the problem of finding the minimal length curve $y(x)$ between two points $(-1, y_0)$ and $(1, y_0)$, subject to the constraint that

$$G\{y\} = \int_{-1}^{1} y\sqrt{1 + y'^2}\, dx = A,$$

for some constant A.

(a) Show that this problem is equivalent to the problem of finding the shape of a hanging cable of length L.

(b) Write the form of the solution. You need not go through the entire derivation, but do please explain how to derive the constants that appear in the solution.

(c) Explain why this “shortest path” is not a straight line.

[6 marks]

5. Consider the functional

$$I\{y, z\} = \int_{x_0}^{x_1} y^2 + z^2 \, dx,$$

subject to the constraint

$$y' = z - y.$$

(a) What type of constraint do we have?

(b) Write down a functional we could optimize to minimize I subject to the constraint.

(c) Determine the Euler-Lagrange equations for y and z.

(d) Solve the equations to find the form of the extremal curve of I under the constraint.

[12 marks]

6. Consider minimizing the functional

$$F\{y\} = \int y''^2 + y^2 \, dx.$$

   (a) Find and simplify the DE that the extremal curves must satisfy using the Euler-Poisson equations.

   (b) Introduce a new variable $u = y'$ using a Lagrange multiplier to remove second order terms from the integral, and derive the resulting Euler-Lagrange DEs.

   (c) Do the two approaches give the same result? Justify.

[12 marks]