Examination in School of Mathematical Sciences

Semester 2, 2010

006128	Variational Methods and Optimal Control
	APP MATH 3010

Official Reading Time:	10 mins
Writing Time:	180 mins
Total Duration:	190 mins

NUMBER OF QUESTIONS: ??   TOTAL MARKS: ??

Instructions

- Answer ALL questions.
- Begin each answer on a new page.
- Examination materials must not be removed from the examination room.

Materials

- 1 Blue books are provided.
- Calculators are NOT permitted.
- 2 double sided pages of handwritten notes are allowed.

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO.

1. **Solutions:**

problem	autonomous	degenerate	dependent on y
$F\{y\} = \int_a^b \frac{\sqrt{1+y'^2}}{x}\,dx$	no	no	no
$F\{y\} = \int_a^b \sin(y)y' + xy'\,dx$	no	yes	yes
$F\{y\} = \int_a^b \sin(xy')\,dx$	no	no	no
$F\{y\} = \int_a^b yy'(1+y')\,dx$	yes	no	yes

[12 marks]

2. **Solution:**

The Euler-Lagrange equations are

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = \frac{d}{dx}\left[3xy'^2\right] = 3y'^2 + 6xy'y'' = 3y'(2xy'' + y') = 0$$

[2 marks]

So either $y' = 0$ or $2xy'' + y' = 0$. The first case has solution

$$y = k,$$

for constant k. Clearly this solution doesn't fit the end points, and so we can exlude it from consideration. [1 marks]

The second DE can be tackled by changing varible to $u = y'$ and dividing by $2x^{1/2}$ to get

$$x^{1/2}u' + x^{-1/2}u/2 = 0,$$

which is just

$$\frac{d}{dx}x^{1/2}u = 0.$$

The solution is

$$x^{1/2}u = const.$$

Substituting $y' = u$, and rearranging we get

$$\begin{aligned} y' &= const \times x^{-1/2} \\ y &= c_1 x^{1/2} + c_2. \end{aligned}$$

NB: interestingly, the solution $y = k$ is a special case of this extremal. [4 marks]

Substituting the end points, we get $c_2 = 0$ and $c_1 = 1$, so the solution is

$$y = x^{1/2}.$$

[1 marks]

alternative solution:

Note that the function f in the integral doesn't depend on y, so we can write the Euler-Lagrange equations

$$\frac{\partial f}{\partial y'} = 3xy'^2 = const$$

The above has two cases: (1) the constant is zero, so $y' = 0$ and hence

$$y = const$$

and (2)

$$y' = \frac{const}{\sqrt{x}},$$

which we can integrate to get

$$y' = c_1\sqrt{x} + c_2,$$

as before.

3. **Solutions:**

(a) There are two dependent variables (x, y) and so two Euler-Lagrange equations:

$$\begin{aligned} \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} - \frac{\partial f}{\partial x} &= 0 \\ \frac{d}{dt}\frac{\partial f}{\partial \dot{y}} - \frac{\partial f}{\partial y} &= 0. \end{aligned}$$

[2 marks]

(b) The E-L equations for this case will be

$$\begin{aligned} \frac{d}{dt}\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} &= 0 \\ \frac{d}{dt}\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} &= 0. \end{aligned}$$

We can directly integrate to get

$$\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = c_1 \tag{1}$$

$$\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = c_2. \tag{2}$$

Take the case where $\dot{y} \neq 0$ and divide (**??**) by (**??**) (we can solve by dividing in the other direction if $\dot{y} = 0$), and we get (by the chain rule)

$$\frac{dx}{dt} \Big/ \frac{dy}{dt} = \frac{dx}{dy} = c.$$

The result is a straight line. [4 marks]

(c) The functional described in the previous problem gives the distance of a path $(x(t), y(t))$, and so finding the shortest path involves minimising this integral, and as expected the solution is a straight line.

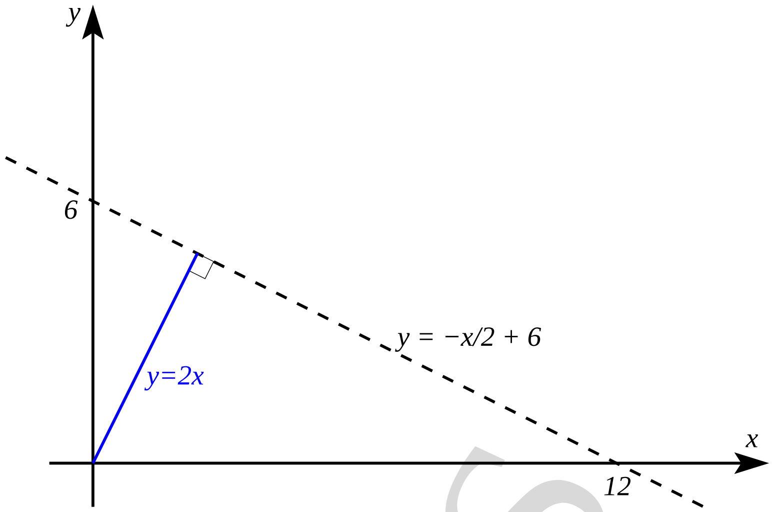
[1 marks]

The transversality condition will require the extremal to meet the line $y = -x/2 + 6$ at a right angle. The slope of the curve is $-1/2$, and so the slope of the extremal is 2, and as it passes through the origin it must have equation

$$y = 2x$$

[2 marks]

The following figure shows the result:



[1 marks]

4. **Solution:**

(a) The arclength of the curve can be written

$$F\{y\} = \int_{-1}^{1} \sqrt{1+y'^2}\, dx.$$

Including the isoperimetric constraint via a Lagrange multiplier μ we seek extremals of the functional

$$J\{y\} = \int_{-1}^{1} \sqrt{1+y'^2} + \mu y\sqrt{1+y'^2}\, dx.$$

Take $\lambda = 1/\mu$ and we get

$$\lambda J\{y\} = \int_{-1}^{-1} (\lambda + y)\sqrt{1+y'^2}\, dx.$$

which is exactly the same as the functional used in finding the shape of a hanging wire of length L, and so the result will be a catenary.

[2 marks]

(b) As noted above, the form of the solution is a catenary of fixed length, which has solution

$$y(x) = c_1 \cosh((x - c_2)/c_1) - \lambda,$$

However, as the solution end points are symmetric, i.e., $(-1, y_0)$ and $(1, y_0)$, the constant $c_2 = 0$ and the solution takes the form

$$y(x) = c_1 \cosh(x/c_1),$$

where in the standard catenary problem λ and c_1 are chosen to solve $y_0 = c_1 \cosh(1/c_1) - \lambda$, and to fix the length to be L. However, here, the constraint it that $G\{y\} = A$, which gives

$$\begin{aligned} G\{y\} &= \int_{-1}^{1} y\sqrt{1+y'^2}\, dx \\ &= \int_{-1}^{1} (c_1 \cosh(x/c_1) - \lambda) \cosh(1/c_1)\, dx \\ &= c_1 + \frac{c_1^2}{2} \sinh(2/c_1) - 2\lambda c_1 \sinh(1/c_1), \end{aligned}$$

and so we solve this on conjunction with the end-point condition. The solution is obtained numerically.

[2 marks]

(c) Without the constraint, the geodesic would obviously be a straight line. Mathematically, we can see that the constraint changes the objective function we seek to optimize, and hence it is not surprising that the shape of the curve changes. More intuitively though, the constraint limits the types of curves that can be considered as viable alternatives. Clearly, straight lines are excluded by this constraint (in all but limiting cases).

[2 marks]

Note: In general there is a reciprocal relationship between optimization objective and isoperimetric constraint. We can usually exchange their roles (provided $\lambda \neq 0$).

5. **Solutions:**

(a) The constraint is a **non-holonomic**. [1 marks]

(b) The Lagrange multiplier is $\lambda(x)$, and the new functional is

$$J\{y, z\} = \int_{x_0}^{x_1} y^2 + z^2 + \lambda(y' - z + y)\, dx.$$

[1 marks]

(c) The Euler-Lagrange equations are

$$\begin{aligned} \frac{d}{dx}\frac{\partial h}{\partial y'} - \frac{\partial h}{\partial y} &= 0, \\ \frac{d}{dx}\frac{\partial h}{\partial z'} - \frac{\partial h}{\partial z} &= 0, \\ \frac{d}{dx}\frac{\partial h}{\partial \lambda'} - \frac{\partial h}{\partial \lambda} &= 0, \end{aligned}$$

[2 marks]

which give

$$\begin{aligned} 2y + \lambda - \lambda' &= 0, & (3) \\ 2z - \lambda &= 0, & (4) \\ y' - z + y &= 0. & (5) \end{aligned}$$

[2 marks]

NB: I will not ommit marks if (**??**) is absent as it is just the original constraint.

(d) Equation (**??**) gives

$$\lambda = 2z,$$

substitute into the first equation and we get

$$2y + 2z - 2z' = 0.$$

Differentiate and rearrange and we get

$$y' = -z' + z'',$$

and we substitute these two into into the last equation to get

$$y' - z + y = -z' + z'' - z - z + z' = 0,$$

which simplifies to the linear homogenous ODE

$$z'' - 2z = 0.$$

This has solutions

$$z = c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x}.$$

We also know

$$y = -z + z' = c_1(-1 + \sqrt{2})e^{\sqrt{2}x} + c_2(-1 - \sqrt{2})e^{-\sqrt{2}x},$$

which we can see satisfies the constraints.
We would need boundary conditions to determine the values of c_1 and c_2. [6 marks]

6. **Solutions**

(a) The Euler-Poisson equation is

$$\frac{d^2}{dx^2}\frac{\partial f}{\partial y''} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{\partial f}{\partial y} = 2\frac{d^2}{dx^2}y'' + 2y = 0.$$

The resulting simplified DE is

$$y^{(4)} + y = 0.$$

[3 marks]

(b) If we introduce the new variable u, with the non-holonomic constraint $u = y'$, then we can rewrite $y'' = u'$ in the integral, and include the constrait via a Lagrange multiplier function $\lambda(x)$ to get a new functional to minimize

$$J\{y, u, \lambda\} = \int h(y, u, \lambda, y', u', \lambda')\, dx = \int u'^2 + y^2 + \lambda(y' - u)\, dx.$$

[2 marks]

The Euler-Lagrange equations are

$$\begin{aligned} \frac{d}{dx}\frac{\partial h}{\partial y'} - \frac{\partial h}{\partial y} &= 0, \\ \frac{d}{dx}\frac{\partial h}{\partial u'} - \frac{\partial h}{\partial u} &= 0, \\ \frac{d}{dx}\frac{\partial h}{\partial \lambda'} - \frac{\partial h}{\partial \lambda} &= 0, \end{aligned}$$

which gives the three linear ODEs

$$\begin{aligned} \lambda' - 2y &= 0, & (6)\\ 2u'' + \lambda &= 0, & (7)\\ y' - u &= 0. & (8) \end{aligned}$$

[3 marks]

(c) The three DEs can be simplified as follows: take the second derivative of (**??**) to obtain

$$y^{(3)} - u'' = 0.$$

Now substitute u'' derived from (**??**) to get

$$y^{(3)} + \lambda/2 = 0.$$

Differentiate this once more

$$y^{(4)} + \lambda'/2 = 0,$$

and substitute λ' derived from (**??**) and we get

$$y^{(4)} + y = 0.$$

[3 marks]

So **yes**, the solutions are the same.

[1 marks]

Please turn over for page 10