Variational Methods and Optimal Control
Extra Questions

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This contains some extra questions related to the Calculus of Variations and its use in Optimal Control. The questions are generally harder than those in class exercises, but show a wider range of applications and ideas than we have time to cover in lectures or tutorials.

The questions here are not explicitly examinable, but if you understand these, you should be very well prepared for the exam.

1. **The Chain Rule:** Given $u = x^2 + 2y$, where

$$\begin{aligned} x(r,t) &= r\sin(t). \\ y(r,t) &= \sin^2(t), \end{aligned}$$

determine the values of

$$\frac{\partial u}{\partial r} \quad \text{and} \quad \frac{\partial u}{\partial t}.$$

Solution:

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \\ &= 2x\sin(t) + 2\times 0, \\ &= 2r\sin^2(t), \\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} \\ &= 2xr\cos(t) + 4\sin(t)\cos(t), \\ &= 2(r^2+2)\sin(t)\cos(t). \end{aligned}$$

2. **Taylor Series:** The "line-picking" problem is the problem of estimating the distribution of the lengths of randomly chosen lines within some region. For instance, start with a circlar region of radius R, and choose two points (independently and uniformly at random within the region). Then the probabiliy density function for the length of this line takes the form $[1,2]$:

$$g(t) = \frac{4t}{\pi R^2} \cos^{-1}\left(\frac{t}{2R}\right) - \frac{2t^2}{\pi R^3}\sqrt{1-\frac{t^2}{4R^2}}$$

This is a rather complicated function. Use Taylor series (around $t = 0$) to find a simple, 3rd order approximation for the density.

Solution: It would be quite hard work to calculate all of the derivatives of this function, and then insert them into a series. A simpler approach is to the Taylor series of the sub-components of the function:

$$\begin{aligned} \cos^{-1}(x) &= \frac{1}{2}\pi - x - \frac{1}{6}x^3 + \cdots \\ \sqrt{1-x^2} &= 1 - \frac{x^2}{2} - \frac{x^4}{8} + \cdots . \end{aligned}$$

to derive

$$\begin{aligned} g(t) &= \frac{4t}{\pi R^2} \cos^{-1}\left(\frac{t}{2R}\right) - \frac{2t^2}{\pi R^3}\sqrt{1-\frac{t^2}{4R^2}} \\ &= \frac{4t}{\pi R^2}\left(\frac{1}{2}\pi - \frac{t}{2R}\right) - \frac{2t^2}{\pi R^3} + O(t^4) \\ &= \frac{2t}{R^2} - \frac{4t^2}{\pi R^3} + O(t^4) \end{aligned} \tag{1}$$

from which we can immediately see that

$$\begin{aligned} g^{(0)}(0) &= 0, \\ g^{(1)}(0) &= \frac{2}{R^2}, \\ g^{(2)}(0) &= -\frac{8}{\pi R^3}, \\ g^{(3)}(0) &= 0. \end{aligned}$$

Notice that because the 3rd order term is zero, the approximation is quite good.

3. **Minimizing functions:** A chain hanging between two pylons takes the shape commonly called a *catenary*, which has mathematical form

$$y = c_1 \cosh\left(\frac{x-c_2}{c_1}\right) - \lambda,$$

where the constants λ, $c_1 > 0$ and c_2 are determined by the length L of the chain, and the end conditions, i.e., the heights of the poles $y(x_0) = x_0$ and $y(x_1) = x_1$.

Assume we have calculated λ, c_1 and c_2 for a chain of length L, and given pylon heights, derive the minimal height of the chain. Be careful to consider all possible cases, and to argue that it is a minimum, not just a stationary point.

Solutions:

$$\begin{aligned} y &= c_1 \cosh\left(\frac{x-c_2}{c_1}\right) - \lambda \\ y' &= \sinh\left(\frac{x-c_2}{c_1}\right), \end{aligned}$$

which is a non-decreasing function, with a zero at $x = c_2$. So y is decreasing to the left of $x = c_2$, and increasing to the right. Hence there are three possible locations for the mimimum – the two edges, or the stationary points $y' = 0$.

$$x_{\min} = \begin{cases} x_0, & \text{if } c_2 \leq x_0, \\ c_2, & \text{if } x_0 \leq c_2 \leq x_1, \\ x_1, & \text{if } c_2 \geq x_1. \end{cases} \tag{2}$$

In these cases we get

$$y_{\min} = \begin{cases} y_0, & \text{if } c_2 \leq x_0, \\ c_1 - \lambda, & \text{if } x_0 \leq c_2 \leq x_1, \\ y_1, & \text{if } c_2 \geq x_1. \end{cases} \tag{3}$$

The points y_0 and y_1 can easily be seen to be minima (when appropriate) because of the increasing or decreasing nature of $y(x)$ over the interval (x_0, x_1) in each case, respectively. The point $y_{\min} = c_1 - \lambda$ can be argued to be a minimum (when appropriate) physically, or we can calculate the second dereviative:

$$y'' = \frac{1}{c_1} \cosh\left(\frac{x-c_2}{c_1}\right),$$

which is positive for all x, meaning that the point $y' = 0$ must be a minimum.

4. **Seashell morphology:** [3] Many seashells take the form of a logarithmic spiral, and a natural question arises, why? The most important consideration is that they must grow incrementally, and as such they need to be able to add to a shell as they develop, without rebuilding the entire thing. A simple cone would, however, provide this facility, so why build a "cone wrapped around a logarithmic spiral?"

   One answer has been postulated for planispiral shells[1] that they are maximizing a kind of structural strength associated with flat springs (like watch springs). The proposed function to minimize (with $x - y$ written as a function of ϕ) is

$$J\{x, y\} = \frac{\alpha}{2} \int_0^{\Phi} \left[x^2 + y^2 - (x\dot{y} - \dot{x}y) \right] e^{-2\alpha\phi} \, d\phi.$$

   Write and solve the resulting Euler-Lagrange equations.

   Solution: There will be two Euler-Lagrange equations, one in x and the other in y, in the form

$$\begin{aligned} \frac{d}{d\phi} \frac{\partial f}{\partial \dot{x}} - \frac{\partial f}{\partial x} &= 0 \\ \frac{d}{d\phi} \frac{\partial f}{\partial \dot{y}} - \frac{\partial f}{\partial y} &= 0 \end{aligned}$$

   resulting firstly in the equation

$$\begin{aligned} \frac{d}{d\phi} \left[y e^{-2\alpha\phi} \right] - 2x e^{-2\alpha\phi} + \dot{y} e^{-2\alpha\phi} &= 0 \\ -2\alpha y e^{-2\alpha\phi} + \dot{y} e^{-2\alpha\phi} - 2x e^{-2\alpha\phi} + \dot{y} e^{-2\alpha\phi} &= 0 \\ -\alpha y + \dot{y} - x &= 0 \\ \dot{y} &= \alpha y - x. \end{aligned}$$

   Combined with the second equation we get

$$\begin{aligned} \dot{y} &= \alpha y - x \\ \dot{x} &= x + \alpha y. \end{aligned}$$

   It is easy to check the equations have solutions

$$\begin{aligned} x &= A e^{\alpha\phi} \cos\phi \\ y &= A e^{\alpha\phi} \sin\phi, \end{aligned}$$

   which is just the parametric form of a logarithmic spiral.

[1] "The Mathematics of Gnomonic Seashells", Chris Illert, Mathematical Biosciences, 63:21-56, 1983

5. **Free-surface of a rotating fluid:** Consider a fluid of density ρ in a cylindrical drum of radius R, which is rotating at angular speed $\omega = \dot{\phi}$, and which has been doing so for long enough that the entire fluid is similarly rotating. Calculate the shape of the free-surface (the interface between the air and fluid), ignoring friction and surface tension.

   Solution: The surface's shape will be rotationally invariant, due to rotational symmetry, and so we shall represent it in cylindrical co-ordinates as a function of radius by $z(r)$.

   Consider the potential energy of a particle at point (r, s, θ). There are two forces applied to the particle

$$\begin{aligned} \text{gravity} &= mg \text{ in the direction -z} \\ \text{centripetal} &= mr\omega^2 \text{ in the direction +r.} \end{aligned}$$

   Thus, we can deduce the form of the potential of the particle to be

$$\begin{aligned} V_{\text{gravity}}(r, s, \theta) &= mgs \\ V_{\text{centripetal}}(r, s, \theta) &= -mr^2\omega^2. \end{aligned}$$

   (noting that potential is defined so that the force in direction x_i is $-\partial V/\partial x_i$). The kinetic energy of the particle is

$$T = \frac{1}{2} m r^2 \omega^2.$$

   To calculate the minimal energy free surface, we take

$$F\{z\} = \int_0^R \int_0^{2\pi} \int_0^z \left[V_{\text{gravity}} + V_{\text{centripetal}} + T \right] r \, ds \, d\theta \, dr,$$

   where the extra factor of r in the integral comes from the Jacobian of the transform from Cartesian to cylindrical co-ordinates (think of the affect of integrating around a cylinder of radius r).

   Now, first consider integrating with respect to the height of the particle s, we get

$$\int_0^z mgs + \frac{1}{2} m r^2 \omega^2 \, ds = \frac{m}{2} \left[g z^2 + \omega^2 r^2 \right],$$

   and note that there is no dependence on θ in the integral, so that contributes a simple factor of 2π so that

$$F\{z\} = m\pi \int_0^R \left[g z^2 r + \omega^2 r^3 \right] \, dr,$$

   We seek the shape $z(r)$ that minimizes this functional. Obviously the constant factor $m\pi$ has no affect on the shape of the solution so we ignore it here. The functional is not dependent on r' (and thus is trivially linear in r' and so the Euler-Lagrange are degenerate reducing to

$$2gzr + \omega^2 r^3 = 0.$$

   Obviously $r \neq 0$ except at the center, so we can rearrange this equation to get

$$z = \frac{\omega^2}{2g} r^2,$$

   i.e., the shape of the surface is a paraboloid of revolution.

   Notes:

   - Note that this is only true for the case where the height of the fluid at the center is $z(0) = 0$. If we seek to determine the solution to this problem in general we can control the volume of fluid, and this constraint

$$V\{z\} = 2\pi \int_0^R z r \, dr.$$

     when added to the above (as an isoperimetric constraint with a Lagrange multiplier) will provide a vertical shift in the solution.

- There is a second (perhaps simpler) approach to solving free surface problems. There can be no tangential forces along a free fluid surface, otherwise, particles in the fluid would travel along the surface, changing its shape. So in equilibrium forces must balance.

  The gravitational force (tangential to the surface) on a point of the surface is $mg\sin\theta$ (towards the center of the drum), and the centripetal force is $mr\omega^2\cos\theta$ (towards the rim), where $\tan\theta = dz/dr$. Balancing the forces gives

$$\begin{aligned} mg\sin\theta &= mr\omega^2\cos\theta \\ \tan\theta &= \frac{r}{g}\omega^2 \\ \frac{dz}{dr} &= \frac{r\omega^2}{g} \\ z &= \frac{\omega^2}{2g}r^2 + c, \end{aligned}$$

  as before.

- This is not an academic problem. This solution is actually used to create large parabolic mirrors for use in astronomy. The Large Zenith Telescope in Canada is the largest such telescope with a pool of mercury of diameter of 6m, and about 8.5 revolutions per minute. `http://en.wikipedia.org/wiki/Liquid_mirror`. Such telescopes cost about 1% of the cost of a similar sized conventional mirror, but not surprising can only be pointed straight up.

6. **Generalization:**

Consider the following problem — find the extremal curves of the following functional:

$$F\{y\} = \int_0^1 g(x)\; {y'}^2 \; dx,$$

for twice differentiable function $g(\cdot) > 0$, and $y(0) = 0$ and $y(1) = 0$.

(a) Show the functional is bounded below by zero.

(b) Solve the general problem using the Euler-Lagrange equations.

(c) Solve the specific problem for the following cases:

i. $g(x) = 1/x^a$ for $a > 0$:
$$y = \frac{c}{a+1} x^{a+1} + k.$$

ii. $g(x) = x$:
$$y = c \ln(x) + k.$$

iii. $g(x) = x^a$, for $a > 1$:
$$y = -\frac{c(a-1)}{x^{a-1}} + k.$$

iv. $g(x) = e^{ax}$:
$$y = -\frac{c}{a} e^{-ax} + k.$$

Solution:

(a) As $g(x) > 0$, and ${y'}^2$, the terms inside the integal are never less than zero, so the integral has a lower bound at zero.

(b) The Euler-Lagrange equations are
$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = \frac{d}{dx} 2g(x)y' = 0.$$
So
$$g(x)y' = const,$$
or
$$y = \int \frac{c}{g(x)}\, dx + k.$$

(c) Examples:

i. $g(x) = 1/x^a$ for $a > 0$:
$$y = \frac{c}{a+1} x^{a+1} + k.$$

ii. $g(x) = x$:
$$y = c \ln(x) + k.$$

iii. $g(x) = x^a$, for $a > 1$:
$$y = -\frac{c(a-1)}{x^{a-1}} + k.$$

iv. $g(x) = e^{ax}$:
$$y = -\frac{c}{a} e^{-ax} + k.$$

7. **Brachystochrone for a rotating object: Brachystochrone for a rotating object:** The classic brachystochrone is based on a sliding, frictionless bead. Now compute the shape of a brachystochrone for an object rolling down the curve (for the sake of argument assume it is a spherical marble though the result should be extendable to any rotationally symmetric object), which does not slip, but experiences no frictional losses of energy.

   Solution: As before we will exploit conservation of ernergy which is made up of kinetic and potential energy. We can find the potential energy of an object by taking the mass to be located at its center of mass. For the sake of what follows assume that the object is rotationally symmetric, and so its center of mass is also the center of rotation, so the potential energy term will be the same as for the standard Brachystochrone problem.

   A rolling object has two sorts of kinetic energy:

   - The translational kinetic energy associated with the movement of the center of mass in the direction it is rolling, which is just the standard
   $$T_1 = \frac{1}{2} m v^2.$$
   - The kinetic energy associated with rotation, which is just
   $$T_2 = \frac{1}{2} I \omega^2,$$
   where ω is the angular velocity (measured in radians per second), and I is the moment of inertia (about the axis of rotation). Example moments of inertia are given below for objects of radius r and mass m.

object	I
solid sphere	$\frac{2}{5} mr^2$
thin spherical shell	$\frac{2}{3} mr^2$
solid cylinder	$\frac{1}{2} mr^2$
thin cylindrical hoop	mr^2

   For the marble in question, the kinetic energy is therefore
   $$T_2 = \frac{1}{5} m r^2 \omega^2.$$

   When there is no slippage, the rate of angular rotation ω, and the velocity of the object are directly linked by
   $$v = r\omega.$$

   Thus the total kinetic energy of a rolling object is
   $$T = T_1 + T_2 = \frac{1}{2} m \left[v^2 + I\omega^2 \right] = \frac{1}{2} m v^2 \left[1 + \frac{2}{5} \right] = \frac{7}{10} m v^2.$$

   Note that this is in exactly the same form as the kinetic energy of a sliding bead, but the constant is different. Energy conservation gives the velocity at a point to be
   $$v = \sqrt{\frac{10}{7} \left[\frac{E}{m} - gy \right]},$$

   Now remember the functional of interest for the brachystochrone is
   $$T\{s\} = \int_0^L \frac{ds}{v(s)} = \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{\frac{10}{7}\left[\frac{E}{m} - gy\right]}} dx = \int_{x_0}^{x_1} \sqrt{\frac{1+w'^2}{w}} dx,$$

   where we make the substitution
   $$w = \frac{10}{7} \left[\frac{E}{m} - gy \right],$$

   So the form of the solution will still be a cycloid, as for the classical cycloid, though the constants of integration may differ.

8. **Clothoid or Euler Spiral:** In previous consideration of a bent elastic beam or cantilever (length d), we assumed that distortions from horizontal were small, and that we could therefore approximate the elastic energy as

$$V = \frac{\kappa}{2} \int_0^d y''^2 \, dx,$$

where κ is the flexural rigidity. In the case where the beam is bent beyond the limits of approximation, what shape will it take? In particular, what shape would it take if there is a point force of P on the end, pushing at right angles to the end of the beam (assume the left end point of the beam is at the origin, and clamped horizontal, but the (x, y) position of the right end-point is free).

[Hint: parameterize the shape of the beam by the arclength s (from the origin) and the tangent angle θ (with reference to the horizontal) at each point. NB: this type of parameterization for a particular curve is sometimes called the Whewell equation of the curve.]

What happens if the force is directly downwards?

Solution: Take the left end point of the beam to be the origin. Parameterize the shape of the beam by $\theta(s)$, the tangent angle at arclength s along the beam, i.e., the position of the beam (x, y) as a function of s can be written

$$\begin{aligned} x &= \int_0^s \cos(\theta(t)) \, dt \\ y &= \int_0^s \sin(\theta(t)) \, dt. \end{aligned}$$

If the force was downwards, the energy of the beam is potential $Py(d)$ plus the elastic energy, which depends on the curvature of the beam at each point, given by θ', so the functional of interest is

$$E\{\theta(s)\} = Py(d) + \frac{1}{2} \int_0^d \kappa \theta'^2 \, ds = \int_0^d \frac{\kappa}{2} \theta'^2 + P \sin(\theta) \, ds.$$

However, we take the simpler case here where the force is applied at right angles to the end of the beam, so that the potential energy is $Pd\theta$, so that the functional of interest is

$$E\{\theta(s)\} = \int_0^d \frac{\kappa}{2} \theta'^2 + P\theta \, ds.$$

Note that this is a linear approximation to the previous functional for small deflections θ. The left end point of the beam is at the origin, and clamped horizontal, so

$$\theta(0) = 0.$$

The Euler-Lagrange equation is

$$\begin{aligned} \frac{d}{ds} \frac{\partial f}{\partial \theta'} - \frac{\partial f}{\partial \theta} &= 0 \\ \kappa \frac{d}{ds} \theta' - P\theta &= 0 \\ \kappa \theta'' - P\theta &= 0. \end{aligned} \tag{4}$$

and hence, we get an equation of the form

$$\theta'' = P/\kappa,$$

which has solution

$$\theta = \frac{P}{\kappa} s^2 + a_1 s + a_0,$$

with curvature

$$\theta' = 2\frac{P}{\kappa} s + a_1,$$

i.e., the curvature varies linearly with distance along the beam. Note that equations such as this which relate the curvature to arclength are commonly called Cesàro equations. Taking the initial condition $\theta(0) = 0$ into account we get $a_0 = 0$. The value of θ at the right boundary is free (but s is fixed), and so at the right boundary

$$\left.\frac{\partial f}{\partial \theta'}\right|_{s=d} = \kappa\theta' = 0.$$

so curvature at the right boundary is zero, i.e.,

$$a_1 = -2\frac{P}{\kappa}d.$$

The resulting shape is shown in Figure 1 (a). This solution is called the *Clothoid*, *Spiros* or *Cornu Spiral* or *Euler Spiral*. Figure 1 (b) shows the Euler spiral.

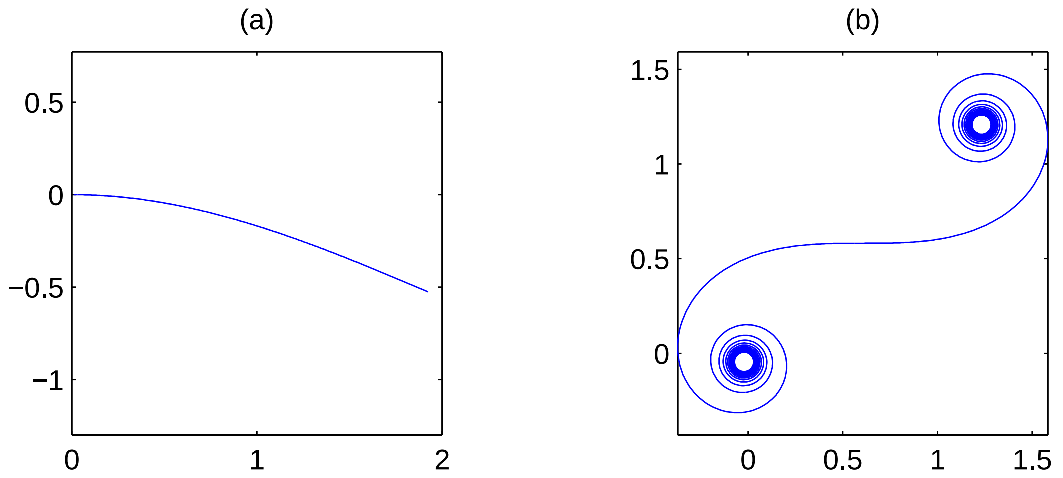


Figure 1: Clothoid of Euler spiral for $P = \kappa = 1$. Figure (a) shows a solution to the cantilever problem for $d = 2$, while figure (b) shows a more general picture of the Euler-Spiral as it is often shown.

The Euler-Spiral has been used to model non-linear splines[2], and the ideal transition curves for railways (see next question).

Note that if we went back to the Euler-Lagrange equations for a downward force we would get

$$\begin{aligned} \frac{d}{ds}\frac{\partial f}{\partial \theta'} - \frac{\partial f}{\partial \theta} &= 0 \\ \kappa\theta'' + P\cos(\theta) &= 0. \end{aligned} \qquad (5)$$

The DE given in (5) is harder to solve, but is reminiscent of the non-linear pendulum DE, which is

$$\phi'' + \omega^2 \sin\phi = 0.$$

In fact we can convert one to the other by a simple change of variables $\phi = \theta + \pi/2$, and $\omega = \sqrt{P/\kappa}$. The non-linear pendulum has solution[3] given by

$$\phi(s) = 2\arcsin\left\{\sin\frac{\phi_0}{2} sn\left[K\left(\sin^2\frac{\phi_0}{2}\right) - \omega t; \sin^2\frac{\phi_0}{2}\right]\right\}.$$

where $K(m)$ is the complete elliptical integral of the first kind, $sn(u;m)$ is the Jacobi elliptic function, and $\phi_0 = \phi(0)$ is the initial value of ϕ (assuming the pendulum starts at rest $\phi'(0) = 0$). These solutions form part of a more general set of curves called the *elastica*.

[2] "Modeling of Curves and Surfaces with MATLAB", Springer Undergraduate Texts in Mathematics and Technology, 2010, Volume 7, Part 2, 201-244, `http://www.springerlink.com/content/x7n001g2752p5136/fulltext.pdf` and "Variational Study of Nonlinear Spline Curves", Lee and Forsythe, SIAM Review, Vol.15, No.1, Jan, 1973, pp.120-133, `http://www.jstor.org/stable/pdfplus/2029293.pdf?acceptTC=true`.

[3] "Exact solution for the nonlinear pendulum", A. Belendez1, C. Pascual, D.I. Mendez, T. Belendez and C. Neipp, Revista Brasileira de Ensino de Fisica, v. 29, n. 4, p. 645-648, (2007), `http://www.sbfisica.org.br/rbef/pdf/070707.pdf`

9. **Railway design:** there are a number of interesting problems in optimal design of railways, for instance in minimizing the cost of a trip. However, here we imagine a railway that must change directions by an angle $\Delta\theta$. Assume the length of the curved track is d, what is the best shape for the curve?

   More precisely, we might aim to minimize curvature in the rail to minimize the centripetal force. Intuitively this would result in the curved segment being a circular arc. However, this results a sudden change in the force at the join between straight and curved sections of track[4]. A better curve will have zero curvature at the end-points, and would minimize the magnitude of the total change in curvature (or its square, which is easier for us to deal with here).

   Solution: The centripetal force on a train at point s along the track will be $F_c(s) = v^2\theta'$. As velocity v is a constant here we shall WLOG set it to be 1. Rather than simply minimize the force, recognize that minimizing the square of the integrated forces will results in the circular arc, i.e., if

$$J\{\theta\} = \int_0^d \theta'^2 - \lambda\theta' \, ds,$$

   where the second term comes from a Lagrange multiplier given the isoperimetric constraint that $\int_0^d \theta' \, ds = \Delta\theta$, then we get the Euler-Lagrange equations

$$\theta'' = 0,$$

   which has solutions

$$\theta = c_1 s + c_2,$$

   i.e., a circular arc.

   However, if we seek to minimize changes in acceleration we need to minimize changes in curvature, and so we get a functional of the form

$$J\{\theta\} = \int_0^d \theta''^2 - \lambda\theta' \, ds,$$

   with corresponding Euler-Poisson equations

$$\theta'''' = 0,$$

   and solution

$$\theta = c_3 s^3 + c_2 s^2 + c_1 s + c_0.$$

   So we see the result is now a clothoid-like curve, with the curvature varying quadratically along the length of the curve.

   Now we choose a set of coordinates such that

$$\begin{aligned} \theta(0) &= 0 \\ \theta(d) &= \Delta\theta, \end{aligned}$$

   and note that we wish curvature to be zero at the boundaries, i.e.,

$$\begin{aligned} \theta'(0) &= 0 \\ \theta'(d) &= 0. \end{aligned}$$

   The boundary conditions at 0 ensure that $c_1 = c_0 = 0$. The boundary conditions at d then give

$$\begin{aligned} c_3 d^3 + c_2 d^2 &= \Delta\theta \\ 3c_3 d^2 + 2c_2 d &= 0. \end{aligned}$$

   The second condition gives $c_2 = -3dc_3/2$. Substituting into the 1st equations

$$c_3 d^3 \left[1 - 3/2\right] = \Delta\theta,$$

   so

$$c_3 = -\frac{2\Delta\theta}{d^3}.$$

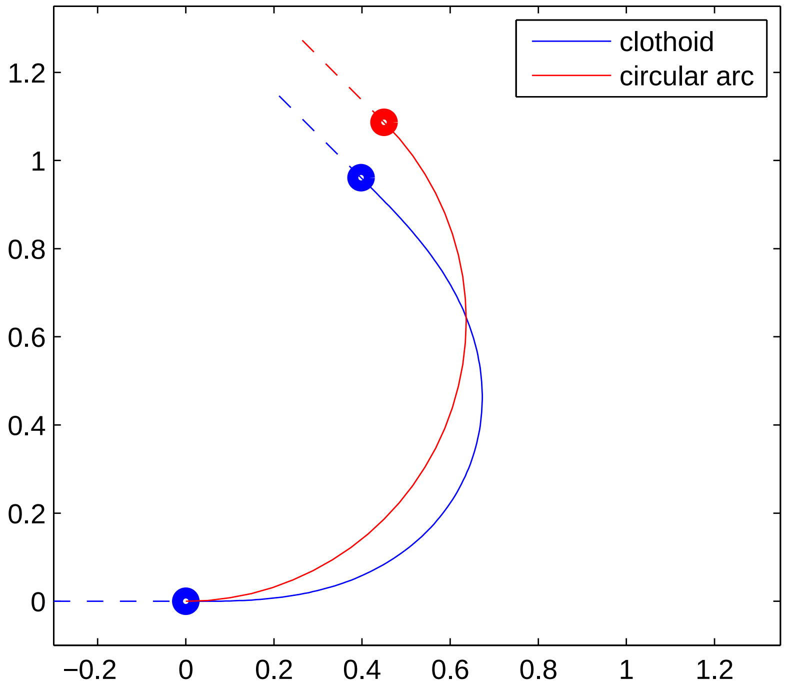


Figure 2: Clothoid transition curve compared to a circular arc.

Figure 2 shows a comparison of the clothoid solution, and the circular arc.

Notice that the maximum curvature of the clothoid is larger than for the circle. Now, if we had a maximum value of θ', then this introduces an inequality constraint, which is either satisfied by the solution above, or we need to have a segment of the curve, where θ' is constant at its maximum value, i.e., it sits on a circular arc for some portion of the transition. In this case, corner conditions (and physical arguments) mean that the Euler-Poisson solution must join the circular arc at a tangent.

Whewell and Cesàro equations provide useful parameterizations for a range of problems, for example here are some curves with simple parameterizations for tangent angle φ, curvature $\varphi' = d\varphi/ds$ and arclength s. Note that the Cesàro equation can be obtained by differentiating the Whewell equation.

Curve	Whewell	Cesàro
Straight Line	$\varphi = c$	$\varphi' = 0$
Circle	$s = r\varphi$	$\varphi' = 1/r$, where r =radius
Catenary	$s = a\tan\varphi$	$\varphi' = \frac{a}{s^2+a^2}$
Log-Spiral	$\varphi = c\log s$	$\varphi' = c/s$
Cornu Spiral	$\varphi = cs^2/2 + k$	$\varphi' = cs$

[4]For instance, if the track is banked to match the centripetal force the banked track and straight track must meet in a smooth curve, and so the transition in the force must be smooth.

10. **Catenary and corner conditions:**

The shape of a hanging chain of length L was presented as the solution of the problem of minimizing potential energy

$$W_p\{y\} = mg \int_{x_0}^{x_1} y\sqrt{1+y'^2}\, dx,$$

under the isoperimetric constraint

$$G\{y\} = \int_{x_0}^{x_1} \sqrt{1+y'^2}\, dx = L.$$

assuming the (given) heights of the pylons $y_i = y(x_i) > 0$.

We determined that the solution to this problem took the form

$$y = c_1 \cosh\left(\frac{x-c_2}{c_1}\right) - \lambda,$$

where the constants λ, c_1 and c_2 are determined by the length L of the chain, and the end conditions, i.e., the heights of the poles $y(x_0) = x_0$ and $y(x_1) = x_1$.

Earlier we calculated the length $L_{\max}$ of a chain that just touched the ground between the two pylons. Now, assume the chain is longer than $L_{\max}$ (but less than $x_1 - x_0 + y_0 + y_1$) and that $y(x) \geq 0$.

Determine the shape of the chain.

Solutions: Nothing has changed about the functional of interest, so given an inequality constraint we know that there are two possibilities:

- the Euler-Lagrange solutions are satisfied and the chain takes the shape of a catenary; or
- the constraint is tight, i.e., $y(x) = 0$, and the chain rests on the ground.

A complete solution for the shape of the chain is made up of segments of these types with "corners" joining them. We know

$$y'' = \frac{1}{c_1}\cosh\left(\frac{x-c_2}{c_1}\right) > 0,$$

for all x, so $y(x)$ is convex, therefore there are only three possible shapes:

(a) the standard catenary, which has zero corners;

(b) a catenary that just touches the ground, which potentially has one corner; or

(c) a catenary with three segments:
   - the left segment has a catenary shape, and is non-increasing;
   - the middle segment is flat (with $y(x) = 0$); and
   - the right segment has catenary shape, and is non-decreasing.

The last case, illustrated in Figure **??** is the one of interest here.

PICTURE

At the corners, which we label x^- and x^+, the chain must be continuous and satisfy the W-E corner conditions. As the value of y is fixed at these corners, the W-E conditions require that the Hamiltonian be continuous at the corners.

The functional being minimized (including the isoperimetric constraint) is

$$F\{y\} = mg \int_{x_0}^{x_1} (y+\lambda)\sqrt{1+y'^2}\, dx,$$

The problem is autonomous, so the corresponding Hamiltonian for the catenary is constant, and given by

$$H = \frac{(y+\lambda)}{\sqrt{1+y'^2}},$$

and for the component where $y = 0$, we have $y' = 0$, and so $H = \lambda$. Given H must be continous at the corners, $H = \lambda$ over the entire curve, and as we already defined (in previous solutions) that $H = c_1$ we get $\lambda = c_1$ (where we know that $c_1 > 0$).

That is the same requirement we had for the curve that had length $L_{\max}$, but now we allow a greater length, and have two additional unknowns, i.e., the locations of the corners. So now we have to solve, for a curve of shape

$$y(x) = \begin{cases} c_1 \left(\cosh\left(\frac{x - c_2^-}{c_1}\right) - 1 \right), & \text{for } x_0 \le x \le x^-, \\ 0, & \text{for } x^- \le x \le x^+, \\ c_1 \left(\cosh\left(\frac{x - c_2^+}{c_1}\right) - 1 \right), & \text{for } x^+ \le x \le x_1, \end{cases}$$

where we now have 5 unknowns c_1, c_2^-, c_2^+, x^- and x^+. In addition to the end-point conditions, and the length constraint, and H continuity at the corners, which is implicity enforced by the form of solution, we have to enforce continuity of y at the corners, i.e., $\lim_{x \to x^-} y(x) = 0$ and $\lim_{x \to x^+} y(x) = 0$. The $\cosh$ function has minimum value 1, and so, this condition implies that $x^- = c^-$ and $x^+ = c^+$. Hence we are left with three conditions and three unknowns:

$$\begin{aligned} y(x_0) &= y_0 \\ y(x_1) &= y_1 \\ L &= (x^+ - x^-) + \int_{x_0}^{x^-} \sqrt{1 + y'^2}\, dx + \int_{x^+}^{x_1} \sqrt{1 + y'^2}\, dx \\ &= (x^+ - x^-) + c_1 \sinh((x^- - x_0)/c_1) + c_1 \sinh((x_1 - x^+)/c_1) \end{aligned}$$

The above can then be solved numerically.

Its is, perhaps, simpler to note that we can use $\cosh^2(x) - \sinh^2(x) = 1$, to get

$$\begin{aligned} c_1 \sinh((x - c_2)/c_1)) &= \text{sign}(x - c_2) \sqrt{c_1^2 \cosh^2((x - c_2)/c_1)) - c_1^2} \\ &= \text{sign}(x - c_2) \sqrt{(y_1 + \lambda)^2 - c_1^2}. \end{aligned}$$

Hence, for instance the left segment is

$$\begin{aligned} L\{y^-\} &= c_1 \sinh((x^- - x_0)/c_1) \\ &= \sqrt{(y_0 + c_1)^2 - c_1^2} \\ &= \sqrt{y_0(y_0 + 2c_1)}, \end{aligned}$$

and that at the left end-point

$$\begin{aligned} y_0 &= c_1 \cosh\left(\frac{x_0 - x^-}{c_1}\right) - c_1 \\ \frac{y_0 + c_1}{c_1} &= \cosh\left(\frac{x_0 - x^-}{c_1}\right) \\ \left(\frac{x_0 - x^-}{c_1}\right) &= \cosh^{-1}\left(\frac{y_0 + c_1}{c_1}\right) \\ x^- &= x_0 - c_1 \cosh^{-1}\left(\frac{y_0 + c_1}{c_1}\right), \end{aligned} \tag{6}$$

and equivalently for the right-hand segment ,and substituting these into the length constraint, we get a non-linear equation in c_1 only

$$L = x_0 + x_1 - c_1 \cosh^{-1}\left(\frac{y_0 + c_1}{c_1}\right) - c_1 \cosh^{-1}\left(\frac{y_1 + c_1}{c_1}\right) + \sqrt{y_0(y_0 + 2c_1)} + \sqrt{y_1(y_1 + 2c_1)},$$

which perhaps simplifies the numerical solution as we can calculate c_1 by a one-dimensional search, and then compute x^- and x^+ directly from (6).

Matlab code is provided below, as are some example results.

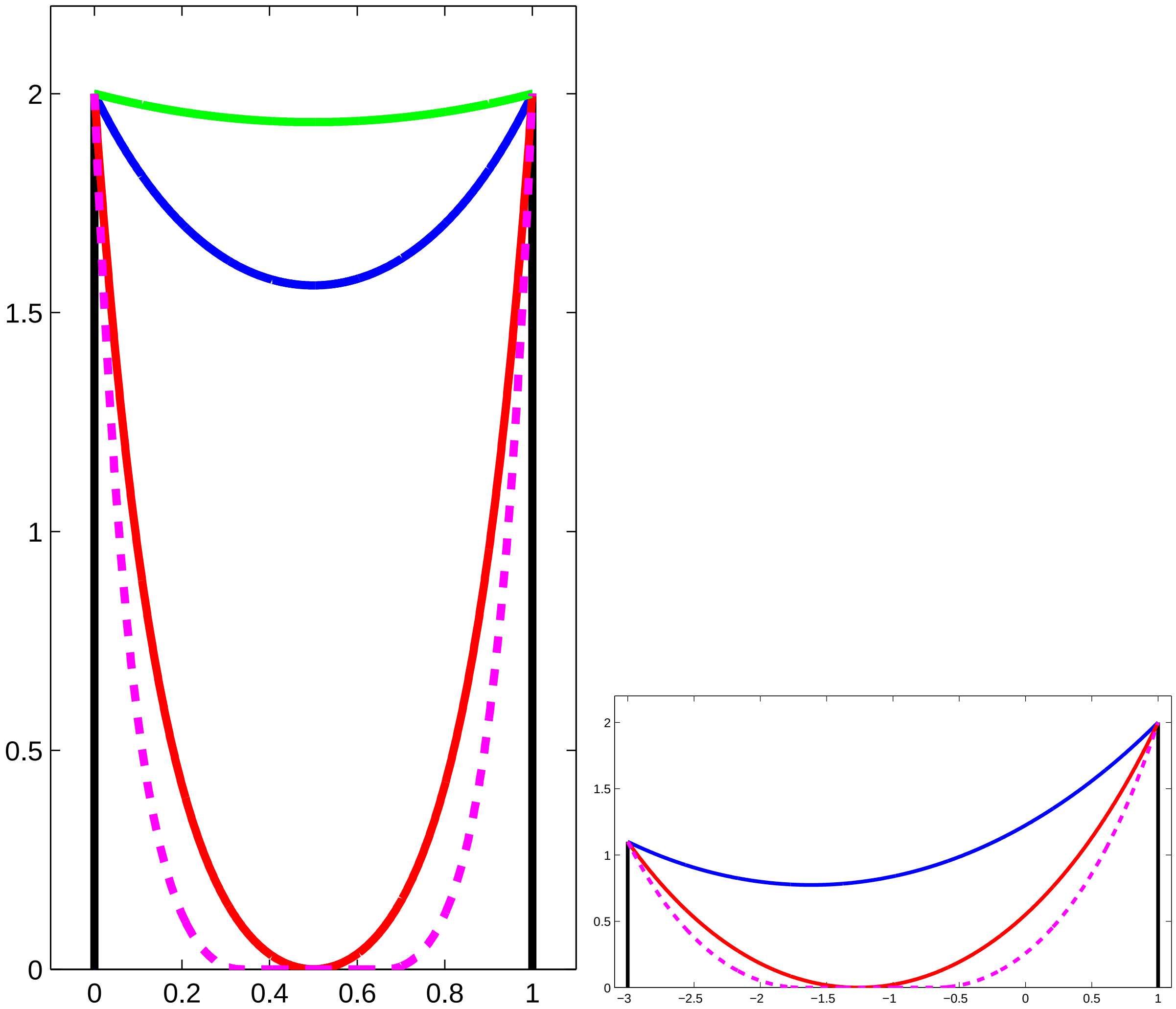


Figure 3: Example catenaries.

Matlab code: for performing estimating catenary parameters is included below.

```
function [x, y, c1, c2_m, c2_p, x_m, x_p, lambda, ...
  Lest, Fest, Lest_check, Fest_check, ...
  f_val, exitflag, output] = catenary_long(n, y_0, y_1, x_0, x_1, L)
%
% file:       catenary_solver_gen.m, (c) 2012 Matthew Roughan
% author:   Matthew Roughan
% email:    matthew.roughan@adelaide.edu.au
%
%
% CATENARY_SOLVER: solves the shape of a hanging chain, which we know will be
%                y = c1*cosh((x-c2)/c1) - lambda
%         with fixed length
%                L = c1.*[ sinh(x_b./c1) - sinh(x_a./c1) ]
%         when the chain is long enough to drag on the ground. In this case it will
%         have three segments (from left to right)
%              -- a downwards part (from x_0 to x-)
%              -- a flat part, y=0, (from x- to x+)
%              -- an upwards part (from x+ to x_1)
%
% INPUTS:
%         n         = number of points at which to calculate the curve
%         y_0       = height of the left pylon
%         y_1       = height of the right pylon
%         x_0       = left pylon position
%         x_1       = right pylon position
%         L         = length of chain
%
% OUTPUTS:
%         [x, y]    = n (x,y) points along the shape of the catenary
%         c1,c2     = constants of integration
%         lambda    = Lagrange multiplier
%         Lest      = estimated length, to be used in debugging
%         Fest      = an estimate of the functional which gives the potential energy of the chain
%         Lest_check = a check based on estimated (x,y) positions (only valid for large n)
%         Fest_check = a check based on estimated (x,y) positions (only valid for large n)
%         [f_val, exitflag, output] = output from the optimization used to find the solution
%


if (y_0 <= 0)
    error(sprintf('y_0=%.3f must be > 0', y_0));
end

if (y_1 <= 0)
    error(sprintf('y_1=%.3f must be > 0', y_1));
end

if (x_1 <= x_0)
  error(sprintf('x_1=%.3f should be > x_0=%.3f', x_1, x_0));
end

[L_max, L_min, c1_max, c2_max, lambda_max] = catenary_max_length(y_0, y_1, x_0, x_1);
if (L <= L_max)
   error(sprintf('You need L=%.3 > L_max=%.3 for it to make sense to use this routine', L, L_max));
end
Lm = (x_1-x_0) + y_1 + y_0;
if (L >= Lm)
  error(sprintf('The chain length L=%.3f is too long even for this routine (max %.3f)', L, Lm));
end

% create a function which we will minimize to find the solution
%    g1 is the left end-point constraint
%    g2 is the right end-point constraint
%    g3 is the length constraint
% a = [c1,c2_m,c2_p]
```

```
g1 = @(a) ( y_0 - a(1)*(cosh( (x_0 - a(2))/a(1) ) -1) ).^2;
g2 = @(a) ( y_1 - a(1)*(cosh( (x_1 - a(3))/a(1) ) -1) ).^2;
g3 = @(a) (L - (a(3)-a(2)) - sqrt(y_0*(y_0+2*a(1))) - sqrt(y_1*(y_1+2*a(1)))).^2;
% g3 = @(a) (L - (a(3)-a(2)) - a(1)*sinh((a(2)-x_0)/a(1)) - a(1)*sinh((x_1-a(3))/a(1)) ).^2;
g = @(a) g1(a) + g2(a) + g3(a);
a_est = [c1_max, (x_0+x_1)/2 - 0.01, (x_0+x_1)/2 + 0.01];
options = optimset('fminsearch');
options = optimset(options, 'MaxFunEvals', 10000);
[a, fval, exitflag, output] = fminsearch(g, a_est, options);
c1 = a(1);
lambda = c1;
c2_m = a(2);
c2_p = a(3);
x_m = a(2);
x_p = a(3);

f_val = [g1(a), g2(a), g3(a)];

% check the length is correct
Lest =  (x_p-x_m) + sqrt(y_0*(y_0+2*c1)) + sqrt(y_1*(y_1+2*c1))
Lest2 = x_1 + x_0 - c1*acosh((y_0+c1)/c1) - c1*acosh((y_1+c1)/c1) + sqrt(y_0*(y_0+2*c1)) + sqrt(y_1*(y_1+2*c1)

%
% now calculate points on the curve
%
x = x_0:(x_1 - x_0)/n:x_1;
k1 = find(x <= x_m);
y(k1) = c1*cosh((x(k1)-c2_m)/c1) - c1;
k2 = find(x > x_m & x < x_p);
y(k2) = 0;
k3 = find(x >= x_p);
y(k3) = c1*cosh((x(k3)-c2_p)/c1) - c1;


% second check of the length
Lest_check = sum(sqrt(diff(x).^2 + diff(y).^2));

Fest = 0;
Fest_check = 0;
```

11. **Discontinuous solutions:** Find the continuous curve y that gives the minimum value of a simple functional of the form

$$J\{y\} = \int_{-a}^{a} x^2 y'^2 \, dx,$$

with $y(-a) = 0$ and $y(a) = 1$. Consider continuity of possible solutions carefully.

Solution: The Euler-Lagrange equation is

$$\frac{d}{dx} 2x^2 y' = 0,$$

so either $y' = 0$ or

$$y' = \frac{c_1}{x^2},$$

which we can integrate easily to get

$$y = -\frac{c_1}{x} + c_2.$$

Taking the second type of solution and inserting the end points we get

$$\begin{aligned} 0 &= \frac{c_1}{a} + c_2 \\ 1 &= -\frac{c_1}{a} + c_2, \end{aligned}$$

which have solutions

$$\begin{aligned} c_1 &= -\frac{a}{2} \\ c_2 &= \frac{1}{2}. \end{aligned}$$

However, note that this curve has a singularity at $x = 0$, and is therefore inadmissible.

The functional clearly has an upper bound on its minimum value of 0, if $y' = 0$. However, if the end points $y(-a) \neq y(a)$, then this solution is not allowed because it cannot be continuous.

However, we can consider a smoothed versions of the curve

$$y = \begin{cases} 0 & \text{where } x < 0, \\ 1 & \text{where } x \geq 0. \end{cases}$$

by interpolating between them with a smooth function within distance ε of the origin. In doing so we can create a curve for which the functional is arbitrarily close to zero. So in fact, there is no minimal curve, only a series of curve closer and closer to a minimum.

Part of the problem lies in the fact that although the function $f = x^2 y'^2$ is continuous, and has two continuous derivatives, it has

$$f_{y'y'} = 0,$$

at $x = 0$. At such points we can have problems in Euler-Lagrange solutions.

Razmadzé defined a family of admissible discontinuous curves D by a set of curves that have a sequence C_n of admissible curves such that

$$\begin{aligned} \lim_{n \to \infty} C_n(x) &= D(x). \\ \lim_{n \to \infty} J\{C_n\} &= J\{D\}, \end{aligned}$$

limiting the number of discontinuities, but allowing for us to use curves of the above type to find solutions that satisfy the Euler-Lagrange equations almost everywhere.

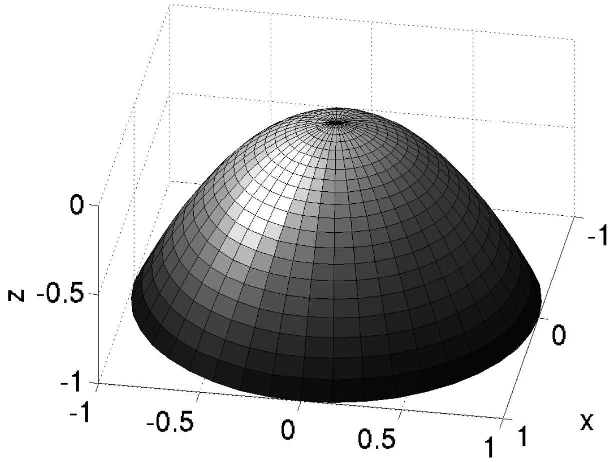
12. **Geodesics:**

Given two points on a circular-paraboloid, find the shortest path between them). That is, find the curve $y(x)$ such that

$$F\{y\} = \int_a^b ds = \int_{x_0}^{x_1} \sqrt{1 + y'^2 + z'^2}\, dx$$

is minimized, subject to fixed end points (x_0, y_0, z_0) and (x_1, y_1, z_1) and the curve lying on the surface

$$z(x, y) = -\alpha(x^2 + y^2)$$



Solution: We will solve for $\alpha = -1$ for convenience. In this case we can write the surface in parametric form as

$$\begin{aligned} x(r,\theta) &= r\cos\theta, \\ y(r,\theta) &= r\sin\theta, \\ z(r,\theta) &= r^2. \end{aligned}$$

We will take the case where we write the geodesic angle as a function of radius (i.e., $\theta(r)$), and then the arclength of a curve on the surface is

$$L\{\theta\} = \int \sqrt{P + 2Q\theta' + R\theta'^2}\, dr,$$

where $theta' = d\theta/dr$ and

$$\begin{aligned} P &= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 \\ &= \cos^2\theta + \sin^2\theta + 4r^2 \\ &= 1 + 4r^2 \\ Q &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta} \\ &= -r\cos\theta\sin\theta + r\sin\theta\cos\theta \\ &= 0 \\ R &= \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 \\ &= r^2\sin^2\theta + r^2\cos^2\theta \\ &= r^2. \end{aligned}$$

That is, we need to find minimal curves of the functional

$$L\{\theta\} = \int \sqrt{1 + 4r^2 + r^2\theta'^2}\, dr.$$

There is no dependence on θ in the integral, therefore the Euler-Lagrange equations simplify to

$$\frac{\partial f}{\partial \theta'} = \frac{r^2\theta'}{\sqrt{1 + 4r^2 + r^2\theta'^2}} = k,$$

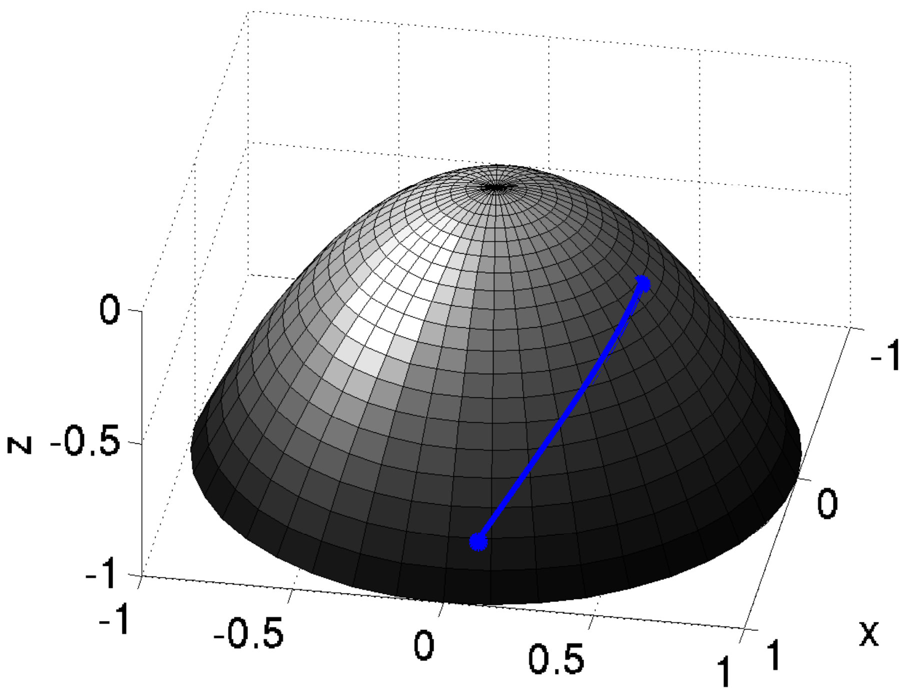
for some constant k. Rearranging the equations we get

$$
\begin{aligned}
r^4\theta'^2 &= k^2(1+4r^2+r^2\theta'^2) \\
r^2(r^2-k^2)\theta'^2 &= k^2(1+4r^2) \\
\theta'^2 &= \frac{k^2(1+4r^2)}{r^2(r^2-k^2)} \\
\theta &= \beta + \int \frac{k}{r}\sqrt{\frac{1+4r^2}{r^2-k^2}}\,dr
\end{aligned}
$$

Now, this integral is not easy, but we can do it using Maple (or any other symbolic manipulation package) to get

$$
\begin{aligned}
\theta &= \beta + \\
&\quad \frac{1}{8k}\Bigg\{ -\ln(2) + 12\,k^2\ln(2) - 8\,k^2\ln\left(1-4\,k^2+8\,r^2+4\,\sqrt{-(1+4\,r^2)(-r^2+k^2)}\right) \\
&\qquad -4\frac{k^2}{\sqrt{-k^2}}\,\ln\left(-\frac{2\,k^2-r^2+4\,r^2k^2-2\,\sqrt{-k^2}\sqrt{-(1+4\,r^2)(-r^2+k^2)}}{r^2}\right)\Bigg\}. \\
&= \beta + \\
&\quad \Bigg\{ -\frac{1}{8k}\ln(2) + \frac{3k}{2}\ln(2) - k\ln\left(1-4\,k^2+8\,r^2+4\,\sqrt{-(1+4\,r^2)(-r^2+k^2)}\right) \\
&\qquad -\frac{i\,\mathrm{sign}(k)}{2}\,\ln\left(-\frac{2\,k^2-r^2+4\,r^2k^2-2\,\sqrt{-k^2}\sqrt{-(1+4\,r^2)(-r^2+k^2)}}{r^2}\right)\Bigg\}.
\end{aligned}
$$

Given start and end points (r_0,θ_0) and (r_1,θ_1), we can use numerical techniques to find integration constants β and k, and thence draw the geodesic, e.g. see



13. **Classification of extrema:** Consider minimizing arc length (as in a geodesic problem)

$$J\{y\} = \int \sqrt{1 + y'^2}\, dx.$$

- Show that Legendre's necessary condition for a local minimum is satisfied.
- Consider the second variation, and show that a geodesic (an extremal for the above functional) will be a local minimum.

Solutions: Legendre's necessary condition requires that $f_{y'y'} > 0$ along the extremal. Now

$$\begin{aligned} f_{y'y'} &= \frac{\partial}{\partial y'} \frac{y'}{\sqrt{1 + y'^2}} \\ &= \frac{1}{\sqrt{1 + y'^2}} - \frac{y'^2}{(1 + y'^2)^{3/2}} \\ &= \frac{(1 + y'^2) - y'^2}{(1 + y'^2)^{3/2}} \\ &= \frac{1}{(1 + y'^2)^{3/2}} \\ &> 0 \end{aligned}$$

which is Legendre's necessary condition.

The second variation is

$$\delta^2 F(\eta, y) = \int_{x_0}^{x_1} \left[\eta^2 f_{yy} + 2\eta\eta' f_{yy'} + \eta'^2 f_{y'y'} \right] dx,$$

where

$$\begin{aligned} f_{yy} &= 0 \\ f_{yy'} &= 0 \\ f_{y'y'} &= \frac{1}{(1 + y'^2)^{3/2}} > 0. \end{aligned}$$

Now the first two terms in the second variation vanish, so that it becomes

$$\delta^2 F(\eta, y) = \int_{x_0}^{x_1} \eta'^2 f_{y'y'}\, dx,$$

where $\eta'^2 \geq 0$ and we have already shown that $f_{y'y'} > 0$, so the second variation must be positive for all perturbations about the extremal, and therefore any extremal of the above functional must be a local minimum.

14. **Yet Another Cantilever Problem:** Going back to the original cantilever problem, we often see the functional written in the form

$$F\{y\} = \int_0^d \frac{EI}{2} y''^2 + \rho y \, dx,$$

where E is the Young's modulus (or the elastic modulus) of the material, and I is the *second moment of area* defined about the line on which we are bending. For instance, given our beam lies in the (x, y) plane, we are effectively bending it around the z axis, so the second moment of area would be defined (at a distance x along the beam) by

$$I(x) = \int_A y^2 \, dA,$$

where A is the cross-sectional area of the beam at x, in the (y, z) plane.

Examples:

cross section shape	area	I
solid rectangular (height h, width w)	hw	$wh^3/12$
solid circular (radius r)	πr^2	$\pi r^4/4$
ring (inner radius r_1, outer radius r_2)	$\pi(r_2^2 - r_1^2)$	$\frac{\pi(r_2^4 - r_1^4)}{4}$

Given in this form, we can solve problems where the shape of the beam varies over its length, i.e. I is a function of x.

Solve the following problem: imagine we want to determine how far the wing of a plane will deflect. Consider a jet plane, with a delta-shaped wings, so that they form triangles with the base fixed horizontally to the side of the plane. The length of the wing is d, and the width at the base is b. Given this form, take $EI(x)$ to be in the form

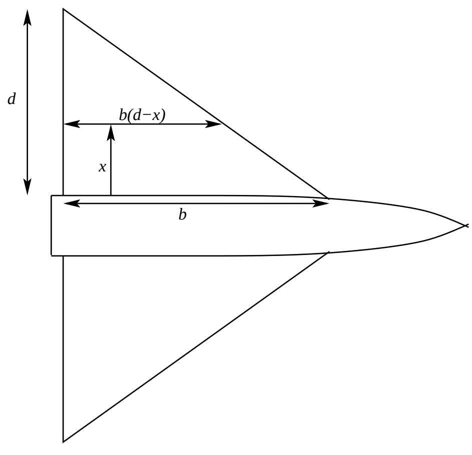
$$EI(x) = B(d - x),$$

for some constant B. Similarly, assume the lift generated by the wing is proportional to its width at distance x, and so the force on the wing (upwards) can be written in the form

$$\rho(x) = C(d - x),$$

for some constant C. Determine the shape of the wing.

(a) Convair Delta Dart



(b) Wing dimensions

Figure 4: Delta-winged aircraft..

Solution: The energy function is in the form

$$F\{y\} = \int_0^d \frac{EI}{2} y''^2 + \rho y \, dx = \int_0^d \frac{B(d-x)}{2} y''^2 + C(d-x) y \, dx,$$

The Euler-Poisson equation is

$$\begin{aligned}
\frac{d^2}{dx^2}\frac{\partial f}{\partial y''} + \frac{\partial f}{\partial y} &= \frac{d^2}{dx^2}\left[B(d-x)y''\right] + C(d-x) \\
&= B\frac{d}{dx}\left[(d-x)y''' - y''\right] + C(d-x) \\
&= B\left[(d-x)y'''' - 2y'''\right] + C(d-x) \\
&= 0.
\end{aligned}$$

Dividing by $B(d-x)$ and taking $z = y'''$ we get

$$z' - \frac{2}{d-x} z = -\frac{C}{B}.$$

Solve the homogenous form of this DE using an integrating factor

$$M(x) = \exp\left[\int -\frac{2}{d-x}\, dx\right] = \exp\left[2\ln(d-x)\right] = (d-x)^2.$$

We get

$$\frac{d}{dx}\left[Mz\right] = -\frac{C}{B}M(x) = -\frac{C}{B}(d-x)^2.$$

Integrating we get

$$Mz = \frac{C}{3B}(d-x)^3 + k_1,$$

or

$$z = \frac{C}{3B}(d-x) + k_1(d-x)^{-2}.$$

However, note that a solution with a pole at d is not acceptible (in fact the natural boundary condition $y'''(d) = z(d) = 0$ ensures this), so that $k_1 = 0$.

Integrating $z = y'''$ to get y we get

$$\begin{aligned}
y''' &= \frac{C}{3B}(d-x) \\
y'' &= -\frac{C}{6B}(d-x)^2 + k_2
\end{aligned}$$

Note the natural boundary conditions $y''(d) = 0$, so $k_2 = 0$, and we integrate again

$$\begin{aligned}
y'' &= -\frac{C}{6B}(d-x)^2 \\
y' &= \frac{C}{18B}(d-x)^3 + k_3
\end{aligned}$$

Now, the boundary condition $y'(0)$ allows us to fix $k_3 = -\frac{Cd^3}{18B}$, and we integrate once more to get

$$\begin{aligned}
y' &= \frac{C}{18B}(d-x)^3 - \frac{Cd^3}{18B} \\
y &= -\frac{C}{72B}(d-x)^4 - \frac{Cd^3}{18B}x + k_4
\end{aligned}$$

and the natural boundary condition $y(0) = 0$ allows us to set $k_4 = \frac{Cd^4}{72B}$ so the final solution is

$$y(x) = -\frac{C}{72B}(d-x)^4 - \frac{Cd^3}{18B}x + \frac{Cd^4}{72B}.$$

The maximum distortion of the wing, at its tip, is

$$y(d) = -\frac{Cd^4}{18B} + \frac{Cd^4}{72B} = -\frac{3Cd^4}{72B}.$$

The solution is interesting, particularly as we can see the curvature of the wing is

$$y'' = -\frac{C}{6B}(d-x)^2.$$

Note though, that the wing breadth is $\propto (d-x)$ so the total stress on the material is proportional to $(d-x)$, i.e., it is largest near the base of the wing.

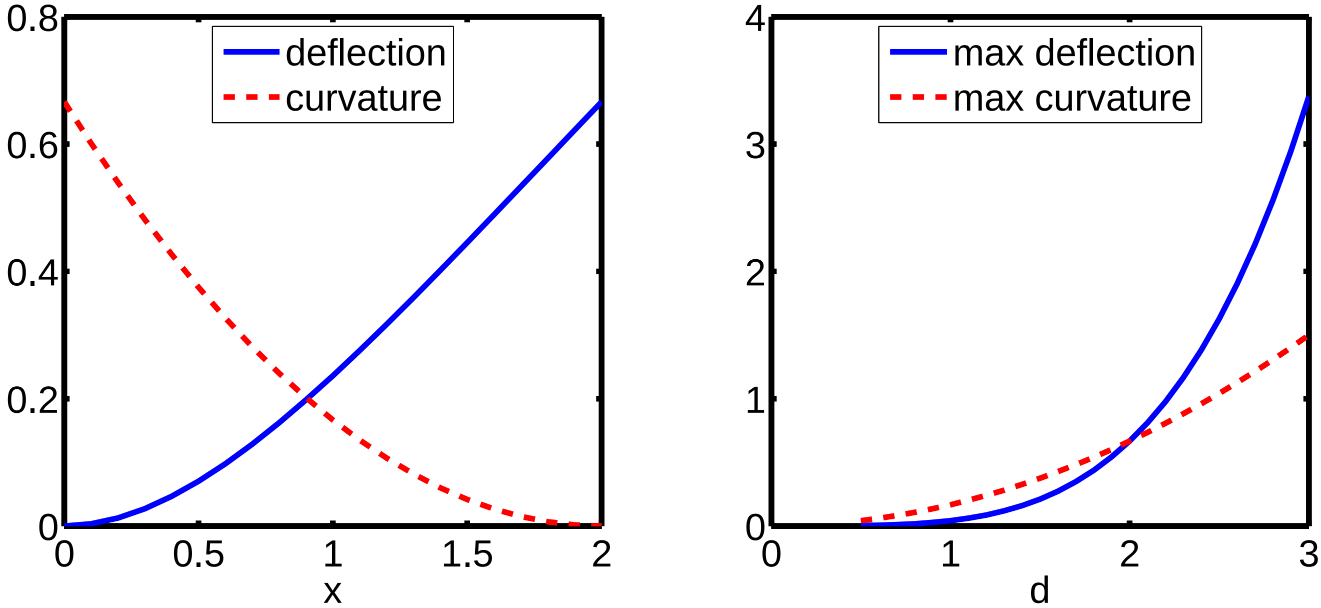


Figure 5: Results for $C = B = 1$ and $d = 2$.

15. **Isoperimetric constraints:** Consider the problem of finding the minimal length curve between two points (x_0, y_0) and (x_1, y_1), subject to the constraint that

$$G\{y\} = \int_{x_0}^{x_1} y\sqrt{1+y'^2}\,dx = L,$$

for some constant L. Find the shape of the extremals.

Solution: Including the isoperimetric constraint via a Lagrange multiplier μ we seek extremals of the functional

$$H\{y\} = \int_{x_0}^{x_1} \sqrt{1+y'^2} + \mu y\sqrt{1+y'^2}\,dx.$$

Take $\lambda = 1/\mu$ and we get

$$\lambda H\{y\} = \int_{x_0}^{x_1} (\lambda + y)\sqrt{1+y'^2}\,dx.$$

which is exactly the same as the functional used in finding the shape of a hanging wire of length L, and so the result will be a catenary.

Note: In general there is a reciprocal relationship between optimization objective and isoperimetric constraint. We can usually exchange their roles (provided $\lambda \neq 0$).

16. **Isoperimetric constraints:** For the functional

$$J\{y\} = \int_0^1 (yy')^2 \, dx$$

subject to $y(0) = 1$, and $y(1) = 2$, and

$$\int_0^1 y^2 \, dx = L$$

(a) Show for $L = 3$ that the extremal takes the form

$$y(x) = \sqrt{4 - 3(x-1)^2}.$$

(b) For $L = 7/3$ show there exists a linear extremal.

(c) For $L = 5/2$ show the problem admits a solution with Lagrange multiplier $\lambda = 0$, and fund the extremal corresponding to this value.

Solution: The problem can be written using a Lagrange multiplier as one of finding the extremals of

$$F\{y\} = \int_0^1 (yy')^2 + \lambda y^2 \, dx.$$

This is an autonomous problem, so the Hamiltonian will be constant, i.e.,

$$H = y' \frac{\partial f}{\partial y'} - f = (yy')^2 - \lambda y^2 = \left[y'^2 - \lambda \right] y^2 = c, \tag{7}$$

for some constant c. There are two classes of solutions to these equations:

(a) If $c \neq 0$, then $y \neq 0$, and therefore $y > 0$ over the interval.
If $\lambda \neq 0$, then we can rearrange (7) to get

$$\begin{aligned} y'^2 &= \frac{c + \lambda y^2}{y^2} \\ y' &= \pm \frac{\sqrt{c + \lambda y^2}}{y} \\ \pm \frac{y}{\sqrt{c + \lambda y^2}} dy &= dx \\ \pm \frac{1}{\lambda} \sqrt{c + \lambda y^2} &= x + k. \end{aligned}$$

Rearranging to get y as a function of x we get

$$y = \sqrt{\frac{\lambda^2 (x+k)^2 - c}{\lambda}},$$

where we can take the positive square root because we know $y > 0$.
The proposed solution:

$$y(x) = \sqrt{4 - 3(x-1)^2}.$$

is in this form, and satisfies the end-point constraints $y(0) = 1$, and $y(1) = 2$, and

$$\int_0^1 y^2 \, dx = \int_0^1 4 - 3(x-1)^2 \, dx = \left[4x - (x-1)^3 \right]_0^1 = [4-1]_0^1 = 3,$$

so this solution satisfies the isoperimetric constraint as well.

(b) If $c = 0$, then the solution requires either $y = 0$, which does not satisfy the end-point conditions, or $y' = \pm\sqrt{\lambda}$, or

$$y = \pm\sqrt{\lambda}x + k,$$

for constant k. In order to satisfy the end-point conditions, we would require

$$\begin{aligned} y(0) &= 1 \quad \Rightarrow k = 1 \\ y(1) &= 2 \quad \Rightarrow \sqrt{\lambda} + k = 2, \end{aligned}$$

so

$$y = x + 1,$$

and $\lambda = 1$. In this case, the integral

$$\int_0^1 y^2\,dx = \int_0^1 (x+1)^2\,dx = \left[\frac{(x+1)^3}{3}\right]_0^1 = \frac{8-1}{3} = \frac{7}{3}.$$

so this solution is only viable if $L = 7/3$.

We know for this case that $y' = 0$ and so the integral $J\{y\} = 0$, which is the smallest possible value, so this solution is a global minimum.

(c) If $c \neq 0$, and we consider the case $\lambda = 0$, then then we can rearrange (7) to get

$$\begin{aligned} yy' &= k \\ \frac{y^2}{2} &= kx + m \\ y &= \pm\sqrt{ax+b}, \end{aligned}$$

for constants $a = 2k$ and $b = 2m$. Solving for the end points we get

$$\begin{aligned} y(0) &= 1 \quad \Rightarrow b = 1 \\ y(1) &= 2 \quad \Rightarrow \sqrt{a+b} = 2, \end{aligned}$$

so

$$y = \sqrt{3x+1}.$$

In this case, the integral

$$\int_0^1 y^2\,dx = \int_0^1 (3x+1)\,dx = \left[\frac{3}{2}x^2 + x\right]_0^1 = \frac{5}{2}.$$

so this solution is only viable if $L = 5/2$.

We know for this case that $yy' = k = a/2 = 3/2$ and so the integral $J\{y\} = k/2 = 3/4$.

17. **Multiple isoperimetric constraints:**

Find the extremals of

$$J\{y\} = \int_0^1 {y'}^2 \, dx,$$

Subject to

$$I_1\{y\} = \int_0^1 y \, dx = 2,$$
$$I_2\{y\} = \int_0^1 xy \, dx = 1/2,$$

and $y(0) = y(1) = 0$.

Solution: We solve for two constraints using two Lagrange multipliers, i.e., find extremals of

$$F\{y\} = \int_0^1 {y'}^2 + \lambda_1 y + \lambda_2 xy \, dx,$$

subject to the end-point conditions. The Euler-Lagrange equations are therefore

$$2y'' = \lambda_1 + \lambda_2 x,$$

which has solutions

$$y = \frac{\lambda_1 x^2}{4} + \frac{\lambda_2 x^3}{12} + c_1 x + c_2,$$

for constants c_1 and c_2. The boundary condition $y(0) = 0$ implies that $c_2 = 0$, and the boundary condition $y(1) = 0$ implies

$$c_1 = -\frac{\lambda_1}{4} - \frac{\lambda_2}{12}.$$

The 1st isoperimetric constraint is

$$\int_0^1 y \, dx = \int_0^1 \frac{\lambda_1 x^2}{4} + \frac{\lambda_2 x^3}{12} + c_1 x \, dx = \left[\frac{\lambda_1 x^3}{12} + \frac{\lambda_2 x^4}{48} + \frac{c_1 x^2}{2} \right]_0^1 = \frac{\lambda_1}{12} + \frac{\lambda_2}{48} + \frac{c_1}{2} = 2,$$

and the 2nd isoperimetric constraint is

$$\int_0^1 xy \, dx = \int_0^1 \frac{\lambda_1 x^3}{4} + \frac{\lambda_2 x^4}{12} + c_1 x^2 \, dx = \left[\frac{\lambda_1 x^4}{16} + \frac{\lambda_2 x^5}{60} + \frac{c_1 x^3}{3} \right]_0^1 = \frac{\lambda_1}{16} + \frac{\lambda_2}{60} + \frac{c_1}{3} = 1/2.$$

These give us three linear equations for c_1, λ_1 and λ_2 which have solutions $c_1 = 42$, $\lambda_1 = 408$ and $\lambda_2 = -720$, which gives the extremal

$$y = 60x^3 - 102x^2 + 42x.$$

18. **Application to statistics:** One of the classic methods for estimation in statistics is called the maximum entropy estimator. The maximum entropy principle is an extension of Laplace's principle of insufficient reason, which in essence says we should not assume things that are not supported by evidence. For instance, in probability, unless we have reason to suspect otherwise, we would assume events are equally likely, e.g., the probability of heads coming up on a coin toss is 1/2.

    Maximum entropy extends this by noting that if we maximize the (Shannon) entropy of a probability distribution constrained by the facts we know about the distribution, we will derive the estimate of that distribution which makes the least assumptions about the distribution that aren't supported by the data.

    The Shannon entropy of a distribution with one continuous variable is defined to be

$$H\{p\} = -\int p(x) \ln p(x) \, dx,$$

    where $p(x)$ is the probability density function (PDF) of the distribution, and $p(x) \ln p(x)$ is understood to be zero whenever $p(x) = 0$. PDFs satisfy certain simple properties, most notably:

    - Non-negativity: the PDF $p(x) \geq 0$. However, Jaynes [4], states that "Mathematically, the maximum-entropy distribution has the important property that no possibility is ignored; it assigns positive weight to every situation that is not absolutely excluded by the given information.", so by this argument we can assume for maximum entropy that $p(x) > 0$ for all allowed values of x.
    - Normalization: the PDF integrates to one, i.e.,

$$\int p(x) \, dx = 1,$$

    Moreover, we often assume we know something about the distribution, for instance, its mean. That is,

$$\int x p(x) \, dx = \mu,$$

    for some known constant μ.

    Use the above to calculate the continuous, maximum entropy distribution for a non-negative random variable with known mean μ.

    Solution: The problem is one of maximizing a functional with two isoperimetric constraints, for a function p defined on $[0, \infty)$. We incorporate these into the problem using two Lagrange multipliers λ_1 and λ_2. Also, *a priori* we only know that the random variable is non-negative, so we don't know the support of $p(x)$. Assume that it is some interval $[S, T]$, then we seek to maximize:

$$F\{p\} = \int_0^T -p(x) \ln p(x) + \lambda_1 p(x) + \lambda_2 x p(x) \, dx.$$

    Note that, although by Jaynes, we should take this integral over all allowed values of x, i.e., the interval $[0, \infty]$, we have not shown in class that such a problem would be a viable fixed-end points problem, so we shall instead condider the problem with a free right end point.

    The Euler-Lagrange equation will be

$$\begin{aligned}
\frac{d}{dx}\frac{\partial f}{\partial p'} - \frac{\partial f}{\partial p} &= 0 \\
-1 - \ln p(x) + \lambda_1 + \lambda_2 x &= 0 \\
\ln p(x) &= \lambda_1 + \lambda_2 x - 1 \\
p(x) &= \exp\big(-1 + \lambda_1 + \lambda_2 x \big) \\
&= A \exp\big(\lambda_2 x\big). \qquad (8)
\end{aligned}$$

The natural boundary conditions will require

$$\begin{aligned}
\frac{\partial f}{\partial p'} &= 0, \\
p' \frac{\partial f}{\partial p'} - f &= 0,
\end{aligned}$$

at the end-point $x = T$. However $\partial f / \partial p' = 0$ so the first condition is trivial, and the second condition becomes

$$\begin{aligned}
-f\big|_T &= 0 \\
p(T)\ln p(T) - \lambda_1 p(T) - \lambda_2 T p(T) &= 0 \\
p(T)\big(\ln p(T) - \lambda_1 - \lambda_2 T\big) &= 0.
\end{aligned}$$

There are two ways to satisfy this equation:

- $p(T) = 0$, or
- $p(T) = \exp\big(\lambda_1 + \lambda_2 T\big)$,

but we already know the form of $p(x)$ in (8), using the Euler-Lagrange equations, and from this know that the second condition cannot be satisfied.

The only way that the first condition can hold given (8) is if were to take the trivial solution $p(x) = 0$ everywhere (which violates the normalization condition), or if $\lambda_2 < 0$, in the limit $T \to \infty$. Thus we can justify taking limits from 0 to ∞.

From the normaization constraint, and $\lambda_2 < 0$, we get

$$\begin{aligned}
\int_0^\infty p(x)\, dx &= 1 \\
\frac{-A}{\lambda_2} &= 1 \\
A &= -\lambda_2,
\end{aligned}$$

and from the constraint on the mean

$$\begin{aligned}
\int_0^\infty x p(x)\, dx &= \mu \\
-\lambda_2 \int_0^\infty x e^{\lambda_2 x}\, dx &= \mu 1 \\
-\int_0^\infty e^{\lambda_2 x}\, dx &= \mu \\
\frac{-1}{\lambda_2} &= \mu \\
\lambda_2 &= -1/\mu,
\end{aligned}$$

so the final solution is the Exponential distribution:

$$p(x) = \frac{1}{\mu} e^{-x/\mu}.$$

Properly, we have not eliminated a function $p(x)$ with corners, or regions that are zero (relying on Jaynes principle), but it should be obvious from the form of the solution to the E-L equation, that these are not possible for a continous function $p(x)$.

Remarks: Maximum entropy is a general principle, and can used to derive other cases, for instance:

- If the random variable has a non-zero probability of taking values in some interval $[a, b]$ then the maximum entropy distribution is the Uniform distribution on that interval.
- If the random variable support over $(-\infty, \infty)$ and has known mean and variance then the maximum entropy distribution is the Gaussian distribution.
- In general, Boltzman showed that if the random variable X had a number of known quanties expressed as expectations

  $$a_i = E[f_i(X)] = \int f_i(x)p(x)\,dx,$$

  and if there is a possible $p(x)$, which satisfies these conditions with positive support over the interval over which X is defined, then the PDF has the following shape:

  $$p(x) = A\exp\left(\sum_i \lambda_i f_i(x)\right),$$

  and this theorem should not seem too hard to prove at this point.

19. **Classification of extrema:** Show that if y satisfies the Euler-Lagrange equations associated with the integral

$$J\{y\} = \int_{x_0}^{x_1} p^2 y'^2 + q^2 y^2 \, dx,$$

where $p(x)$ and $q(x)$ are known functions, then J has the value

$$J\{y\} = \left[p^2 y y' \right]_{x_0}^{x_1}.$$

Solution: The Euler-Lagrange equation will be

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0,$$

which in this case gives

$$\frac{d}{dx}\left[p^2 y' \right] - q^2 y = 0$$

In other words

$$q^2 y^2 = y \frac{d}{dx}\left[p^2 y' \right].$$

Substitute this into the integral and integrating the second term by parts we get

$$\begin{aligned} J\{y\} &= \int_{x_0}^{x_1} p^2 y'^2 + q^2 y^2 \, dx \\ &= \int_{x_0}^{x_1} p^2 y'^2 + y \frac{d}{dx}\left[p^2 y' \right] \, dx \\ &= \int_{x_0}^{x_1} p^2 y'^2 \, dx + \left[y p^2 y' \right]_{x_0}^{x_1} - \int_{x_0}^{x_1} y' p^2 y' \, dx \\ &= \left[p^2 y y' \right]_{x_0}^{x_1}. \end{aligned}$$

Note: following a similar argument for some arbitrary function η we get

$$\begin{aligned} \int_{x_0}^{x_1} p^2 y' \eta' + q^2 y \eta \, dx &= \int_{x_0}^{x_1} p^2 y' \eta' + \eta \frac{d}{dx}\left[p^2 y' \right] \, dx \\ &= \int_{x_0}^{x_1} p^2 y' \eta' \, dx + \left[\eta p^2 y' \right]_{x_0}^{x_1} - \int_{x_0}^{x_1} \eta' p^2 y' \, dx \\ &= \left[p^2 \eta y' \right]_{x_0}^{x_1}. \end{aligned}$$

If we insist that $\eta(x_0) = \eta(x_1) = 0$, then the above integral is zero, and so

$$\begin{aligned} J\{y + \varepsilon\eta\} &= \int_{x_0}^{x_1} p^2 (y' + \varepsilon\eta')^2 + q^2 (y + \varepsilon\eta)^2 \, dx \\ &= \int_{x_0}^{x_1} p^2 y'^2 + q^2 y^2 \, dx + \varepsilon \int_{x_0}^{x_1} p^2 y' \eta' + q^2 y \eta \, dx + \varepsilon^2 \int_{x_0}^{x_1} p^2 \eta'^2 + q^2 \eta^2 \, dx \\ &= J\{y\} + \varepsilon^2 \int_{x_0}^{x_1} p^2 \eta'^2 + q^2 \eta^2 \, dx \\ &\geq J\{y\} \end{aligned}$$

and hence the extremals are automatically local minima.

20. **Dynamic Systems:** Consider a *flyball* or *centrifugal governor* as shown in Figure 6. A governor is used as part of an engine to control its speed. The governor spins around its axis at a rate determined by the engine, but as it spins faster, the balls are pushed outwards by centrifugal force, and this raises them, activiting some control mechanism to slow down the engine. Thus they control its speed.

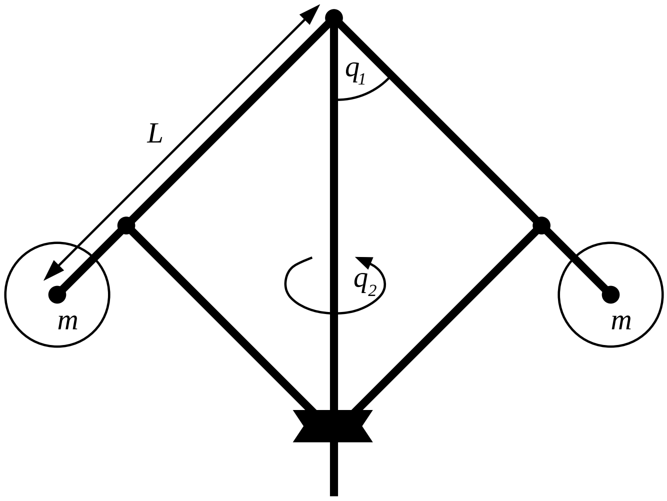


Figure 6: Flyball governor.

Take two generalized coordinates: the first q_1 representing the angle of the upper arms to the upright, and the second q_2 representing the angle of the plane of the arms to some vertical reference plane. We shall assume that the arms are light and do not significantly affect the dynamics of the system, and that the masses are point masses. We shall also ignore the force needed to control the engine.

(a) Use these coordinates to write the Lagrangian for the system.

(b) From this Lagrangian, derive a set of equations of motion.

(c) Determine what simple symmetries apply, and from this derive conservation laws for the system.

Solution:

(a) The Lagrangian is $L = T - V$ for kinetic energy T and potential V. The potential energy comes from the height of the balls, which will be

$$y = L,$$

above their minimum height. There are two balls so the potential is

$$V = 2mgy = 2mgL(1 - \cos q_1).$$

The kinetic energy comes from two components, the rotation around the vertical axis (at rate $\dot{q}_2$), and the upwards/downwards motion of the balls, which we represent as circular motion around the joint between the arms and the top, i.e., it has rate $\dot{q}_1$. Circular motion at rate ω, and at radius r has kinetic energy $\frac{1}{2}mr^2\omega^2$, so the two components of motion here induce kinetic energy

$$T = T_1 + T_2 = mL^2 \left[\dot{q}_2^2 \sin^2 q_1 + \dot{q}_1^2\right].$$

The combined Lagrangian is therefore:

$$L = T - V = m\left[L^2 \dot{q}_2^2 \sin^2 q_1 + L^2 \dot{q}_1^2 - 2gL(1 - \cos q_1)\right].$$

(b) Hamilton's principle of stationary action requires that the paths of motion be stationary (extremals) of the following integral

$$J\{q_1, q_2\} = \int L\,dt.$$

Ignoring the constant factor of m, the Euler-Lagrange equations are

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} &= \frac{d}{dt}\left[2L^2\dot{q}_1\right] - L^2\dot{q}_2^2 2\sin q_1\cos q_1 + 2gL\sin q_1 = 0 \\
\text{or} \quad & L\ddot{q}_1 - L\dot{q}_2^2\sin q_1\cos q_1 + g\sin q_1 = 0 \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} &= \frac{d}{dt}\left[2L^2\dot{q}_2\sin^2 q_1\right] = 0 \\
\text{or} \quad & L^2\dot{q}_2\sin^2 q_1 = const
\end{aligned}$$

(c) The Lagrangian has an explict q_1 term, but no q_2 or t terms, and is thus symmetric under time translations, or rotations about the vertical axis (but not the other axes). The resulting conservation laws are

- Conservation of energy (our model does not include a driving force, or any dissapation due to friction – if it did then energy might not be conserved).
- Conservation of angular momemtum (about the vertical axis).

Interestingly, a careful examination of the second Euler-Lagrange equation shows that angular momentum (about the vertical axis) is conserved.

21. **Electrical Dynamics:** There are often electrical analogues to mechanical systems, the simplest of which is a harmonic oscillator comprised of a capacitor and an inductor as shown in Figure 7.

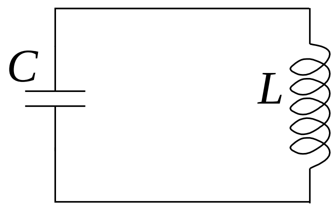


Figure 7: Harmonic oscillator circuit.

Conventionally, this circuit is analysed using Kirchoff's voltage law (that the voltages at the capacity and inductor must be equal), i.e.,

$$V_C = V_L,$$

and Kirchoff's current law (the current through the two components must balance)

$$i_C + i_L = 0.$$

The *constitutive relations* relate current to coltage in the two components by

$$\begin{aligned} V_L(t) &= L\frac{di_L}{dt} \\ i_C(t) &= C\frac{dV_C}{dt}, \end{aligned}$$

where C is the capacitance (measured in *Farads*, and L the inductance (measured in *Henries*) of the circuit. Rearranging these gives a second order DE

$$\ddot{i} + \left(\frac{1}{LC}\right) i = 0, \tag{9}$$

which has simple harmoically oscillating solutions.

However, we can also consider this system as a variational system. Given an appropriate action integral, Hamilton's principle of least action applies here. Treat charge across the capcitor as the dependent variable q, then current i is the rate of change of charge $i = \dot{q}$. The Kirchoff relations allow us to consider only one charge/current.

The analogue of kinetic energy is the energy in the inductor, which for a linear inductor is

$$E_I = \frac{L}{2} i^2,$$

and the analogue of potential energy is the energy stored in the capacitor, which for a linear capacitor is

$$E_C = \frac{1}{2C} q^2.$$

Write an appropriate Lagrangian and show that the Euler-Lagrange equations result in equation (9).

Solution: Taking a Lagrangian of the form

$$L = T - V = E_I - E_C,$$

we get

$$L = \frac{L}{2}\dot{q}^2 - \frac{1}{2C}q^2.$$

The resulting Euler-Lagrange equation is

$$L\ddot{q} + \frac{1}{C}q = 0,$$

and we can easily see that this results in (9).

Notes: The constituitive equations are similar to Newton's equations of motion ($F = ma$), such as we might use in a physical system like the pendulum. Here we show that these results would arise naturally from Hamilton's principle. Much more complicated electronic circuits can be considered in the same way.

22. **Higher-order derivatives and natural boundary conditions:** We have noted that there are two ways of dealing with higher-order derivatives in a functional, for instance

$$J\{y\} = \int f(x, y, y', y'')\, dx.$$

We can tackle it head on using the Euler-Poisson equation

$$\frac{d^2}{dx^2}\frac{\partial f}{\partial y''} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{\partial f}{\partial y} = 0,$$

or we can use a non-holonomic constraint $z = y'$ to introduce the new variable z, via a Lagrange multiplier, i.e.,

$$G\{y\} = \int f(x, y, y', z') + \lambda(x)(z - y')\, dx.$$

and derive the two Euler-Lagrange equations

$$\begin{aligned} \frac{d}{dx}\left[\frac{\partial f}{\partial y'} - \lambda\right] - \frac{\partial f}{\partial y} &= 0 \\ \frac{d}{dx}\frac{\partial f}{\partial z'} - \lambda &= 0. \end{aligned}$$

We showed in lectures that these produced identical extremal curves. Now show that the two approaches generate identical natural boundary conditions (where x_i are the fixed x values at the end points, and y, and y' may vary at the end points).

Solutions: We have already derived the natural boundary conditions for the case with functions of y'' to be

$$\begin{aligned} \left.\frac{\partial f}{\partial y''}\right|_{x_i} &= 0 \\ \left.\frac{\partial f}{\partial y'} - \frac{d}{dx}\frac{\partial f}{\partial y''}\right|_{x_i} &= 0 \end{aligned}$$

Alternatively, we can use the natural boundary conditions for the case with multiple dependent variables, i.e., for dependent variables q_k, and Lagrangian L, we would get

$$\left.\sum_{k=1}^{n} p_k \delta q_k - H\delta t\right|_{x_i} = 0 \text{ where } p_k = \frac{\partial L}{\partial \dot{q}_k} \text{ and } H = \sum_{k=1}^{n} \dot{q}_k p_k - L$$

In the context of this problem $q_1 = y$ and $q_2 = z$, $x = t$ and $L = f + \lambda(z - y')$. The value of x is fixed at the boundary so $\delta x = 0$ in the above, and we can vary δz and δy independently so the components separate and we get the boundary conditions

$$\left.\frac{\partial L}{\partial y'}\right|_{x_i} = 0 \quad \text{and} \quad \left.\frac{\partial L}{\partial z'}\right|_{x_i} = 0.$$

Considering the second condition we get

$$\left.\frac{\partial L}{\partial z'}\right|_{x_i} = \left.\frac{\partial f}{\partial z'}\right|_{x_i} = \left.\frac{\partial f}{\partial y''}\right|_{x_i} = 0$$

which is just the first of the natural boundary conditions above. The first condition gives

$$\left.\frac{\partial L}{\partial y'}\right|_{x_i} = \left.\frac{\partial f}{\partial y'} - \lambda\right|_{x_i} = 0$$

but remember that the Euler-Lagrange equations require that along the curve we have $\lambda = \frac{d}{dx}\frac{\partial f}{\partial z'}$. Making the substitution we get

$$\left.\frac{\partial f}{\partial y'} - \frac{d}{dx}\frac{\partial f}{\partial z'}\right|_{x_i} = \left.\frac{\partial f}{\partial y'} - \frac{d}{dx}\frac{\partial f}{\partial y''}\right|_{x_i} = 0$$

So either approach result in equivalent natural boundary conditions (though it may be that one or the other appears in a form more immediately convenient for solution).

23. **Zermelo's navigation problem (1931):** imagine we are required to pilot a boat (that travels at a constant speed relative to the water) from one side of a river to another, and that the speed of the current in the river depends on the distance from the shore. What is the fastest path across the river? This problem is a specific case of the general Zermelo navigation problem.

    More precisely, consider crossing a river of width d, which we orient along the x-axis as in Figure 8. The river flows from left to right, with speed $v(y)$, that depends only on the distance from the shore y. The boat will have constant speed U with respect to the water (not the shoreline). We have control over the direction in which the boat aims (though its actual direction of movement will be a combination of the boat's and water's velocities). We aim to minimise the transit time from $A \to B$. For simplicity, take A to be at the origin.

    Find the minimal time path for

    - a river with uniform velocity $v(y) = v$
    - a river where the velocity near the centre is faster, following a parabolic law, i.e., $v(y) = Cy(d - y)$.

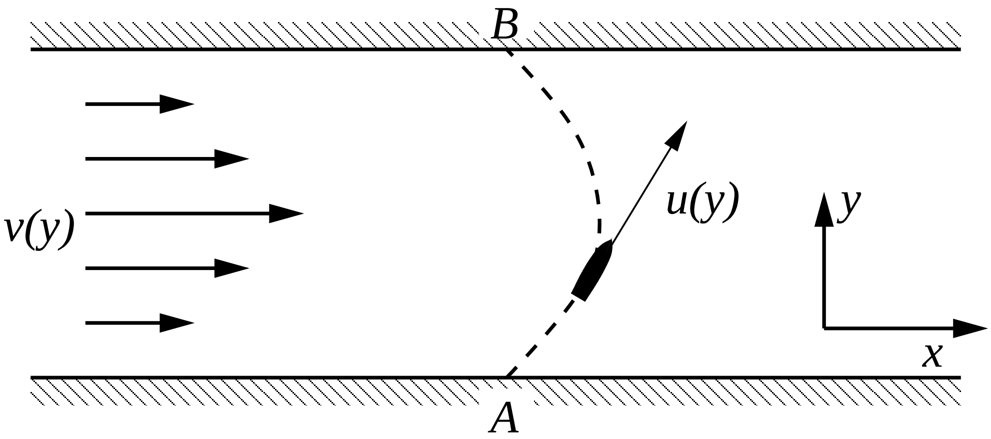


Figure 8: Zermelo's river crossing problem.

Compare the time taken by the optimal path, as compared to direct path (where one steers so as to follow a straight line between A and B) and the path where one simply steers towards the goal.

Solution: The velocity of the boat with respect to the water in the x direction, and y direction respectively will be u_x and u_y respectively. Given the boat will travel with speed U with respect to the water, we know that $u_x^2 + u_y^2 = U^2$.

The boat's velocities with respect to the shore are

$$\begin{aligned} \dot{x} &= u_x + v(y) \\ \dot{y} &= u_y. \end{aligned}$$

The goal to minimize is the time from A to B, so we need to minimize

$$T\{x, y\} = \int_0^d \frac{1}{\dot{y}}\, dy = \int_0^d \frac{1}{u_y}\, dy,$$

i.e., the distance divided by the velocity at each point. However, we also have the isoperimetric constraint that we must end up at the dock at B, and so

$$\Delta x = \int_0^t \dot{x}\, dt = \int_0^d \frac{\dot{x}}{\dot{y}}\, dy = \int_0^d \frac{u_x + v(y)}{u_y}\, dy = 0.$$

where we know that $u_x = \pm\sqrt{U^2 - u_y^2}$. we incorporate the constraint using a Lagrange multiplier to obtain the functional of interest

$$J\{u_y\} = \int_0^d \frac{1 + \lambda\left(\sqrt{U^2 - u_y^2} + v(y)\right)}{u_y}\, dy.$$

The Euler-Lagrange equations are

$$\frac{d}{dy}\frac{\partial f}{\partial \dot{u}_y} - \frac{\partial f}{\partial u_y} = +\lambda(U^2 - u_y^2)^{-1/2} + \frac{1 + \lambda\left(\sqrt{U^2 - u_y^2} + v(y)\right)}{u_y^2} = 0.$$

Multiplying by u_y^2 we get

$$\begin{aligned}
\frac{\lambda u_y^2}{\sqrt{U^2 - u_y^2}} + \lambda\sqrt{U^2 - u_y^2} &= -1 - \lambda v(y) \\
\lambda\frac{u_y^2 + U^2 - u_y^2}{\sqrt{U^2 - u_y^2}} &= -1 - \lambda v(y) \\
\frac{1}{\sqrt{U^2 - u_y^2}} &= -\frac{1 + \lambda v(y)}{U^2\lambda} \\
\sqrt{U^2 - u_y^2} &= -\frac{U^2}{1/\lambda + v(y)} \quad \text{note this means } u_x = -\frac{U^2}{1/\lambda + v(y)} \\
u_y &= \sqrt{U^2 - \frac{U^4}{(1/\lambda + v(y))^2}},
\end{aligned}$$

where we take the positive root because we want to move across the river (in the positive direction). Now that we know $(u_x, u_y$ we can compute the path by integrating, for instance (assuming we start at the origin A),

$$x(y) = \int_0^y \frac{u_x + v}{u_y}\, dy = \int_0^y \frac{-\frac{U^2}{1/\lambda + v(y)} + v}{U\sqrt{1 - \frac{U^2}{(1/\lambda + v(y))^2}}}\, dy = \int_0^y \frac{-U + (1/\lambda + v)v/U}{\sqrt{(1/\lambda + v)^2 - U^2}}\, dy.$$

Now the boundary constraint is that $x(d) = 0$ and this can be used in the above formula to find λ for a given function $v(y)$, and then we can plot the path across the river.

Constant current: For particular cases the above integral may be analytically tractable, for instance, take $v(y) = c$, a constant. Then from the above $u_x = const.$ As $x(y)$ involves an integral over a term that we have set to zero, it is easy to see that the trajectory of the boat must be a straight line across the river, i.e., the boat is oriented so that its drift with the current is compensated exactly by the boats direction. We can calculate the actual speeds by looking at the integral for $x(d) = 0$ (or indeed any $x(y)$ as the trajectory is a straight line), i.e.,

$$x(d) = \int_0^y \frac{u_x + v}{u_y}\, dy = \int_0^d \frac{-U + (1/\lambda + v)v/U}{\sqrt{(1/\lambda + v)^2 - U^2}}\, dy = d\frac{-U + (1/\lambda + v)v/U}{\sqrt{(1/\lambda + v)^2 - U^2}} = 0,$$

which requires that

$$-U + (1/\lambda + c)c/U = 0,$$

which leads to

$$\frac{1}{\lambda} = \frac{U^2}{c} - c,$$

and hence

$$u_x = -c.$$

i.e., the velocity in the x direction (relative to the water) directly counters the current.

The argument of the square root in the denominator must be positive, which leads to the condition for a solution that

$$\begin{aligned}
(1/\lambda + c)^2 - U^2 &> 0 \\
\frac{U^4}{c^2} - U^2 &> 0 \\
U^2(U^2 - c^2) &> 0 \\
U &> c
\end{aligned}$$

The condition makes some sense, as the boats speed along in the x direction cannot be compensated if the speed of the current of the river c is faster than the boats speed U. If the condition is not satisfied, then no trajectory exists that takes the boat across the river to a point directly opposite the start.

Parabolic current: The current in a river is often swifter near the middle, and we might perhaps model this as a parabolic current, i.e.,

$$v(y) = \alpha y(d-y).$$

Note that

$$u_x = -\frac{U^2}{1/\lambda + v(y)} = -\frac{U^2}{1/\lambda + \alpha y(d-y)}.$$

where $1/\lambda$ is a constant. Integrating we get

$$x(d) = \int_0^y \frac{u_x + v}{u_y}\, dy = \int_0^d \frac{-U + (1/\lambda + \alpha y(d-y))\alpha y(d-y)/U}{\sqrt{(1/\lambda + \alpha y(d-y))^2 - U^2}}\, dy = 0,$$

The above isn't easily solved analytically, but can be easily calculated numerically as a function of $1/\lambda$, and hence set to zero. Once $1/\lambda$ is known, the velocities u_x are easily calculated. An example is shown in Figure 9.

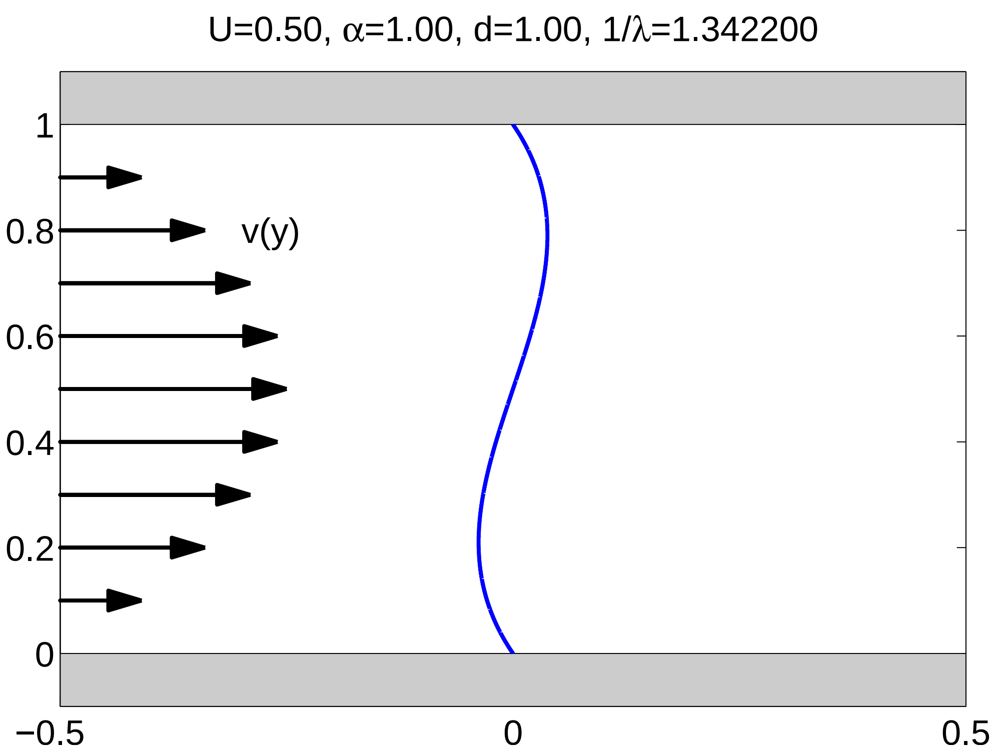


Figure 9: Zermelo's river crossing solution.

Notes: There are many variants of this problem:

- finding the shortest crossing time regardless of where the boat lands;
- finding the shortest time to get to across a bay, or other water body;
- finding shortest times in aeronautical problems (in 3D).

24. **Higher order derivatives via non-holonomic constraints:** Take an autonomous problem with second order derivatives, e.g., find the extremals of

$$J\{y\} = \int F(x, y, y', y'')\, dx.$$

We noted in lectures that this could be solved using a new variable $z = y'$, and rewriting $F(x, y, y', y'') = F(x, y, z, z')$. There is now more than one dependent variable, but no second order derivatives, however, we must also introduce the constraint that $z - y' = 0$ and so we look for extremals of the functional

$$G\{y, z, \lambda\} = \int_a^b f(x, y, z, z') + \lambda(x)(z - y')\, dx.$$

The Euler-Lagrange equations for y and z are

$$\begin{aligned} \frac{d}{dx}\frac{\partial g}{\partial y'} - \frac{\partial g}{\partial y} &= 0 \\ \frac{d}{dx}\frac{\partial g}{\partial z'} - \frac{\partial g}{\partial z} &= 0 \end{aligned}$$

note that $g(x, y, z, z') = f(x, y, z, z') + \lambda(x)(z - y')$ so the E-L equations become

$$\begin{aligned} \frac{d}{dx}\left[-\lambda(x)\right] - \frac{\partial f}{\partial y} &= 0 \\ \frac{d}{dx}\frac{\partial f}{\partial z'} - \frac{\partial f}{\partial z} - \lambda(x) &= 0 \end{aligned}$$

The first Euler-Lagrange equation can be rewritten

$$\frac{d\lambda}{dx} = -\frac{\partial f}{\partial y}$$

Differentiating the second E-L equation WRT x we get

$$\frac{d^2}{dx^2}\frac{\partial f}{\partial z'} - \frac{d}{dx}\frac{\partial f}{\partial z} - \frac{d\lambda}{dx} = 0$$

Note from above that $\lambda' = -f_y$ and that $z = y'$ and $z' = y''$ we get (as before) the Euler-Poisson equation:

$$\frac{d^2}{dx^2}\frac{\partial f}{\partial y''} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{\partial f}{\partial y} = 0$$

Now show that the same happens if we solve the slightly different problem with functional

$$\tilde{G}\{y, z, \lambda\} = \int_a^b f(x, y, y', z') + \lambda(x)(z - y')\, dx.$$

Solution: The Euler-Lagrange equations are the same with respect to $\tilde{g}$, where $\tilde{g}(x, y, z, z') = f(x, y, y', z') + \lambda(x)(z - y')$ so the E-L equations become

$$\begin{aligned} \frac{d}{dx}\left[\frac{\partial f}{\partial y'} - \lambda(x)\right] - \frac{\partial f}{\partial y} &= 0 \\ \frac{d}{dx}\frac{\partial f}{\partial z'} - \lambda(x) &= 0 \end{aligned}$$

Differentiating the second E-L equation WRT x we get

$$\frac{d^2}{dx^2}\frac{\partial f}{\partial z'} - \frac{d\lambda}{dx} = 0$$

The first Euler-Lagrange equation can be rewritten

$$\frac{d\lambda}{dx} = \frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y}$$

Substituting $d\lambda/dx$, $y' = z$ and $y'' = z'$ we get the Euler-Poisson equation:

$$\frac{d^2}{dx^2}\frac{\partial f}{\partial y''} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{\partial f}{\partial y} = 0,$$

as before.

25. **Beltrami Identity in higher order problems:** Take an autonomous problem with second order derivatives, e.g., find the extremals of

$$J\{y\} = \int F(y, y', y'')\, dx,$$

and find the corresponding Beltrami identity.

Solution: The problem can be directly solved using the Euler-Poisson equation, but here we use an alternative.

First convert the problem into one with only first order derivatives by introducing the variable $z = y'$, and enforcing this constraint with a Lagrange multiplier, i.e., the problem becomes: find the extremals of

$$G\{y, z\} = \int F(y, y', z') + \lambda(z - y')\, dx,$$

The problem is automomous, so the Hamiltonian will be constant, i.e.,

$$H = \frac{\partial f}{\partial z'} z' + \frac{\partial f}{\partial y'} y' - f = const,$$

where $f = F(y, z, z') + \lambda(z - y')$. We can expand as follows, replacing $z = y'$ and $z' = y''$

$$\begin{aligned} H &= \frac{\partial f}{\partial z'} z' + \frac{\partial f}{\partial y'} y' - f \\ &= \frac{\partial F}{\partial z'} z' + \frac{\partial F}{\partial y'} y' - \lambda y' - F(y, y', z') - \lambda(z - y') \\ &= \frac{\partial F}{\partial y''} y'' + \frac{\partial F}{\partial y'} y' - \lambda y' - F(y, y', y'') \end{aligned}$$

Now we can find the Lagrange multiplier λ using the Euler-Lagrange equation for z, i.e.,

$$\begin{aligned} \frac{d}{dx}\frac{\partial f}{\partial z'} - \frac{\partial f}{\partial z} &= 0 \\ \frac{d}{dx}\frac{\partial F}{\partial z'} - \lambda &= 0 \\ \lambda &= \frac{d}{dx}\frac{\partial F}{\partial z'} \end{aligned}$$

So the identity becomes

$$\begin{aligned} H &= \frac{\partial F}{\partial y''} y'' + \frac{\partial F}{\partial y'} y' - \lambda y' - F(y, y', y'') \\ &= \frac{\partial F}{\partial y''} y'' + \frac{\partial F}{\partial y'} y' - y' \frac{d}{dx}\frac{\partial F}{\partial y''} - F(y, y', y'') \\ &= const \end{aligned}$$

Remarks: In fact, what we are really doing is deriving higher order versions of the Hamiltonian (and generalized momenta). A fairly general form of this is given in “Noether’s theorem in generalized mechanics”, Dan Anderson, 1973 J. Phys. A: Math. Nucl. Gen. 6 299, `http://iopscience.iop.org/0301-0015/6/3/005`. Converting to our notation, if we take a functional

$$J\{y\} = \int_{x_0}^{x_1} L(x, y, y', \ldots, y^{(n)})\, dx,$$

then the corresponding Hamiltonian is

$$H = \sum_{j=1}^{n} P_j y^{(j)} - L,$$

where the P_j are generalized momenta terms corresponding to each $y^{(j)}$, and are given by

$$P_j = \sum_{i=0}^{n-j} (-1)^i \frac{d^i}{dx^i} \frac{\partial L}{\partial y^{(i+j)}},$$

A simple example is the case when $n = 2$. In this case,

$$\begin{aligned} P_1 &= \frac{\partial L}{\partial y'} - \frac{d}{dx}\frac{\partial L}{\partial y''} \\ P_2 &= \frac{\partial L}{\partial y''} \end{aligned}$$

Note that, as we should expect, these are the terms that appear in the natural boundary conditions for a problem with a second order derivative. P_1 corresponds to free y, and P_2 to free y'. The Hamiltonian (which in natural boundary conditions corresponds to free x at the end point) is then just

$$H = P_1 y' + P_2 y'' - L = y' \left(\frac{\partial L}{\partial y'} - \frac{d}{dx}\frac{\partial L}{\partial y''} \right) + y'' \frac{\partial L}{\partial y''} - L.$$

which we can see is the same as that derived above.

If we extend this to more than one dependent variable, then there will be a series of generalized momenta for each state variable, e.g., for a functional

$$J\{\mathbf{q}\} = \int_{x_0}^{x_1} L(x, \mathbf{q}, \dot{\mathbf{q}}, \ldots, \mathbf{q}^{(n)})\, dx,$$

the generalized momenta terms are given by

$$P_j^{(k)} = \sum_{i=0}^{n-j} (-1)^i \frac{d^i}{dx^i} \frac{\partial L}{\partial q_k^{(i+j)}},$$

and

$$H = \sum_k \sum_{j=1}^{n} P_j q_k^{(j)} - L.$$

The important fact is that these definitions of P_i and H can then be used in variations of Noether's theorem, corner conditions, and in natural boundary conditions corresponding to free end points, though obviously some care needs be taken about the general form of these conditions. For instance, Noether's theorem see "Noether's theorem in generalized mechanics" for the correct version of Noether's theorem.

26. **Optimal Control:** We consider the problem of steering a large ship. We want to change the bearing of the ship from θ_0 to θ_1 in the shortest time possible. The equation describing the bearing of the ship is

$$\ddot{\theta} + \dot{\theta} = F,$$

where F is the rudder setting, which is subject to the restriction $|F| \leq 1$. Essentially, this equation reflects the fact that the faster the ship is turning, $\dot{\theta}$ the less affect the rudder has on the rate of change of turn $\ddot{\theta}$. We can see that when the rate of turn reaches $\dot{\theta} = 1$, then the rudder will no long increase the rate of turn at all, so this is effectively the maximum rate of turn.

For simplicity we take $\theta_0 = 0$, and assume that the ship should be travelling in a straight line before and after the manoeuvre, so that $\dot{\theta} = 0$ at the start and end times.

The optimization problem is find F that minimizes time

$$T = \int_{t_0}^{t_1} 1\, dt = \int_{\theta_0}^{\theta_1} \frac{dt}{d\theta}\, d\theta = \int_{\theta_0}^{\theta_1} \frac{1}{\dot{\theta}}\, d\theta.$$

Solution: We can immediately see that the porblem is likely to be bang-bang type control, but more formally we can introduce the constraint into the optimization objective through a Lagrange multiplier, i.e., minimize

$$J\{F\} = \int_{\theta_0}^{\theta_1} \frac{1}{\dot{\theta}} + \lambda \left(\ddot{\theta} + \dot{\theta} - F \right)\, d\theta.$$

The Euler-Poisson equations will be

$$\begin{aligned} \frac{d^2}{dx^2}\frac{\partial f}{\partial \ddot{\theta}} - \frac{d}{dx}\frac{\partial f}{\partial \dot{\theta}} + \frac{\partial f}{\partial \theta} &= 0 \\ \frac{d^2\lambda}{dx^2} - \frac{d\lambda}{dx} + \frac{d}{dx}\frac{1}{\dot{\theta}^2} &= 0 \\ \frac{d\lambda}{dx} - \lambda + \frac{1}{\dot{\theta}^2} &= const \end{aligned}$$

and

$$\begin{aligned} \frac{d^2}{dx^2}\frac{\partial f}{\partial \ddot{F}} - \frac{d}{dx}\frac{\partial f}{\partial \dot{F}} + \frac{\partial f}{\partial F} &= 0 \\ -\lambda &= 0 \end{aligned}$$

Substituting the second into the first we get

$$\frac{1}{\dot{\theta}^2} = const$$

or

$$\dot{\theta} = const.$$

The only such solution that satisfies the boundary conditions is $\dot{\theta} = 0$, but this is unstatisfactory because then the ship never completes the manoeuvre. On the other hand, the time is minimized by taking $\dot{\theta} = \infty$, which requires infinite turning force, and so is equally unsatisfactory.

Thus, as there is no Euler-Lagrange solution, we conclude that the solution must lie on the boundary of the admissable constrols, i.e., $F = \pm 1$.

Given F, we can solve the state equations. We get

$$\ddot{\theta} + \dot{\theta} = \pm 1,$$

which has solution

$$\theta = c_1 + c_2 e^{-t} \pm t.$$

Take start point $\theta_0 = 0$ and we get $c_2 = -c_1$ so

$$\theta = c_1(1 - e^{-t}) \pm t,$$

and

$$\dot{\theta} = c_1 e^{-t} \pm 1,$$

which also must be zero at the start point, so

$$c_1 = \mp 1,$$

and

$$\begin{aligned} \theta &= \mp(1 - e^{-t}) \pm t \\ \dot{\theta} &= \mp e^{-t} \pm 1 \end{aligned}$$

If we take $\theta_1 > 0$ for the sake of argument, then it is reasonable to assume that for the optimal path $\dot{\theta} > 0$, and so we must start with $F = 1$ and

$$\begin{aligned} \theta &= -(1 - e^{-t}) + t \\ \dot{\theta} &= -e^{-t} + 1 \end{aligned}$$

Assuming a second part of the curve with $F = -1$ we get the curve

$$\theta = c_1 + c_2 e^{-t} - t,$$

where $\theta(t_1) = \theta_1$, but t_1 is unknown. The starting point is the cross-over point $t = t^*$, where $\theta(t^*)$ and $\dot{\theta}(t^*)$ are known, so we can write

$$\theta = c_1 + c_2 e^{-(t-t^*)} - (t - t^*),$$

where

$$\begin{aligned} c_2 &= -1 - \dot{\theta}(t^*) \\ c_1 &= \theta(t^*) - c_2 \end{aligned}$$

or

$$\theta = \theta(t^*) - (1 + \dot{\theta}(t^*)) \left[e^{-(t-t^*)} - 1 \right] - (t - t^*).$$

Given this form, we can again calculate the derivative

$$\dot{\theta} = (1 + \theta(\dot{t}^*)) e^{-(t-t^*)} - 1$$

and if we set this to zero at the right hand boundary and take logs we get

$$\begin{aligned} 0 &= (1 + \dot{\theta}(t^*)) e^{-(t_1 - t^*)} - 1 \\ 1 &= (1 + \dot{\theta}(t^*)) e^{-(t_1 - t^*)} \\ \ln 1 &= \ln(1 + \dot{\theta}(t^*)) + \ln e^{-(t_1 - t^*)} \\ 0 &= \ln(1 + \dot{\theta}(t^*)) - (t_1 - t^*) \\ (t_1 - t^*) &= \ln(1 + \dot{\theta}(t^*)) \end{aligned}$$

where we know that $\theta(\dot{t}^*) = 1 - e^{-t^*}$ so

$$t_1 = t^* + \ln(2 - e^{-t^*}),$$

So we have calculated all constants in terms of t_*, and now just need to know the time of the switch point. We can substitute $1 + \dot{\theta}(t^*) = e^{t_1 - t^*}$ into

$$\begin{aligned} \theta(t_1) &= \theta(t^*) - (1 + \dot{\theta}(t^*)) \left[e^{-(t_1 - t^*)} - 1 \right] - (t_1 - t^*) \\ &= \theta(t^*) - e^{t_1 - t^*} \left[e^{-(t_1 - t^*)} - 1 \right] - (t_1 - t^*) \\ &= \theta(t^*) - \left[1 - e^{t_1 - t^*} \right] - (t_1 - t^*) \end{aligned}$$

and $(t_1 - t^*) = \ln(1 + \dot{\theta}(t^*))$ so

$$\begin{aligned} \theta(t_1) &= \theta(t^*) - \left[1 - e^{t_1 - t^*} \right] - (t_1 - t^*) \\ &= \theta(t^*) + \dot{\theta}(t^*) - \ln(1 + \dot{\theta}(t^*)) \end{aligned}$$

Now up to t^* we already know that

$$\begin{aligned} \theta &= -(1 - e^{-t}) + t \\ \dot{\theta} &= -e^{-t} + 1 \end{aligned}$$

so

$$\begin{aligned} \theta(t_1) &= \theta(t^*) - \left[1 - e^{t_1 - t^*} \right] - (t_1 - t^*) \\ &= -(1 - e^{-t^*}) + t^* + -e^{-t^*} + 1 - \ln(2 - e^{-t^*}) \\ &= t^* - \ln(2 - e^{-t^*}) \end{aligned}$$

which we can solve numerically to get t^*. The solution is illustrated in Figure 10. The curve consists of two phases, the first of positive steering up until time $t^* = 5.691$, and the second, of negative steering until time $t^* = 6.383$. Note that a large part of the trajectory is almost straight as the control limits the maximum rate of turn to $\dot{\theta} \leq 1$, and so for much of the time, the ship is turning at near its maximum rate.

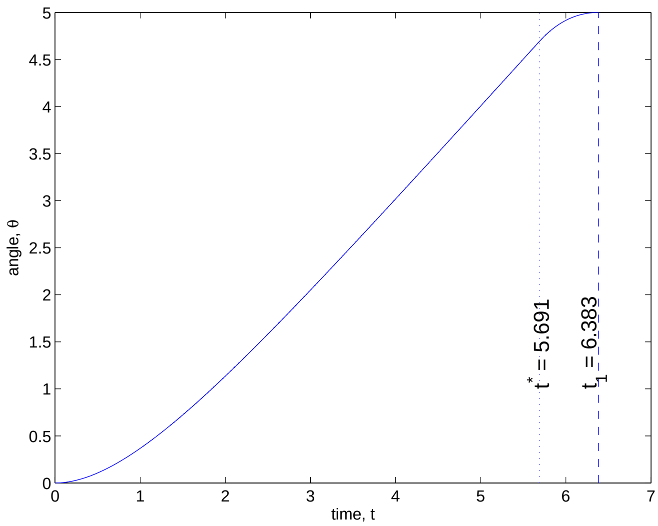


Figure 10: Ship manoeuvre curve for $\theta_0 = 0$ and $\theta_1 = 5$.

27. **Existence:** The question of whether a minimal solution exists is sometimes hihgly non-trivial to answer. This is perhaps best seen in the disarmingly simple sounding *Kakeya needle set* problem.

    The problem is to find the smallest set within which a unit line segment (a needle) can be rotated continuously through 180 degrees so that it returns to its original position with its orientation reversed.

    Solution: The problem sounds like a classical CoV problem similar to Dido's problem. However, int 1927 Besicovitch showed that there was no minimum. Regions can be constructed with arbitrarily small area, but there is no area zero (technically measure zero) region that's is satisfactory, so we can't find a minimum.

    If we restrict the region to be simply connected ()

28. **Geodesics and corners:** In considering geodesics in the plane, we showed that they necessarily consist of straight-line segments. However, we did not show that this was sufficient, and in fact didn't rule out a curve made up of a series of straight lines with corners.

    Use W-E corner conditions to show that the geodesics in the plane must be made up of single line segments without corners.

    Solution:

    The solution is interesting, because it illustrates some of the difficulties in classifying extrema. Point conditions are not sufficient because, for instance, if we had a geodesic from $A \to B$, and from $B \to C$, then a point condition along the curve cannot rule out the curve $A \to B \to C$ being a geodesic between A and C.

    In the special case above, we are saved by the corner conditions preventing us from linking up two different lines, but in general we need a condition that somehow considers the entire curve in order to see whether we have a true minimum (or maximum).

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