Variational Methods & Optimal Control

lecture 11

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Extension 3: several independent variables

When there are several independent variables, e.g., (x,y) and the extremal we wish to find represents, for instance, a surface $z(x,y)$, and f is a function $f(x,y,z(x,y),z_x,z_y)$, then the E-L equation generalizes to give

$$\frac{\partial f}{\partial z} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z_x} - \frac{\partial}{\partial y}\frac{\partial f}{\partial z_y} = 0$$

Several independent variables

Consider a surface minimization problem. We have a surface in 3D that is a function of (x,y), e.g. $z = z(x,y)$ then x and y are both independent variables.

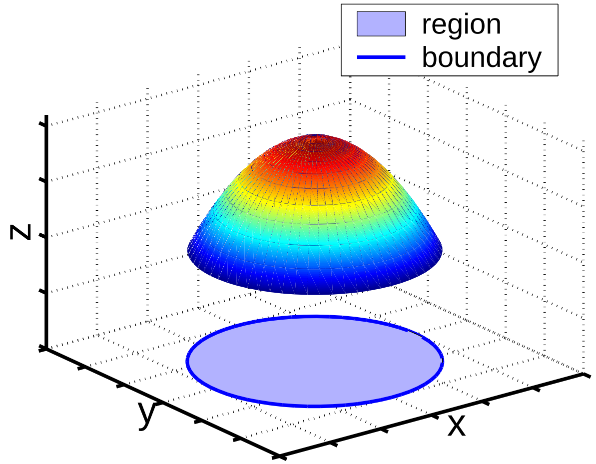
Examples:

- minimal area surfaces
  - soap films and bubbles
  - for construction
- problems of the form, minimize

$$F\{z\} = \iint_\Omega z_x^2 + z_y^2 \, dx\, dy$$

Notation

$$\begin{aligned} \text{region} &= \Omega \\ \text{boundary} &= \delta\Omega \\ \text{surface} &= z(x,y) \end{aligned}$$



Formalisms

Ω is a simply connected, bounded region of $\mathbb{R}^2$

$\delta\Omega$ is the boundary of Ω

$\bar{\Omega} = \Omega \cup \delta\Omega$ is the closure of Ω

$C^2(\bar{\Omega}) = \{z : \bar{\Omega} \to \mathbb{R} \mid z \text{ has 2 continuous derivatives}\}$

$C^2(\delta\Omega) = \{z_0 : \delta\Omega \to \mathbb{R} \mid z_0 \text{ has 2 continuous derivatives}\}$

$\iint_\Omega f(x,y)\,dx\,dy$ is an area integral of f over the region Ω

$\oint_{\delta\Omega} f(x,y)\,dx$ is a contour integral around the boundary $\delta\Omega$.

The problem

Find extremals for the functional

$$F\{z\} = \iint_\Omega f(x,y,z(x,y),z_x,z_y)\,dx\,dy$$

Analogy of fixed end points is a fixed boundary, e.g.

$$z(x,y) = z_0(x,y) \text{ for all } (x,y) \in \delta\Omega$$

for some specified function $z_0 \in C^2(\delta\Omega)$.

Solution

As before we consider perturbations, though in this case they are perturbations to a surface, with fixed edge, e.g.

$$\hat{z}(x,y) = z(x,y) + \varepsilon\eta(x,y)$$

where $\eta(x,y) = 0$ for all $(x,y) \in \delta\Omega$.

Taylor's theorem gives

$$\begin{aligned} & f(x,y,z+\varepsilon\eta, z_x+\varepsilon\eta_x, z_y+\varepsilon\eta_y) \\ = \;& f(x,y,z,z_x,z_y) + \varepsilon\left[\eta\frac{\partial f}{\partial z} + \eta_x\frac{\partial f}{\partial z_x} + \eta_y\frac{\partial f}{\partial z_y}\right] + O(\varepsilon^2) \end{aligned}$$

The First Variation

As before we demand that at an extremal, the First Variation $\delta F(\eta,z) = 0$ for all possible η, and small ε

$$\begin{aligned} \delta F(\eta,z) &= \lim_{\varepsilon\to 0} \frac{F\{z+\varepsilon\eta\} - F\{z\}}{\varepsilon} \\ &= \iint_\Omega \left[\eta\frac{\partial f}{\partial z} + \eta_x\frac{\partial f}{\partial z_x} + \eta_y\frac{\partial f}{\partial z_y}\right] dx\,dy \end{aligned}$$

We next need to do the equivalent of integration by parts, but its a bit more complicated — we need to use Green's theorem.

Green's theorem

One form of Green's theorem states

$$\iint_\Omega \left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \right) dx\, dy = \oint_{\delta\Omega} \phi\, dy - \oint_{\delta\Omega} \psi\, dx$$

for $\phi, \psi : \bar{\Omega} \to \mathbb{R}$ such that ϕ, ψ, ϕ_x and ψ_y are continuous.

This converts an area integral over a region into a line integral around the boundary.

Green's theorem in use

Green's theorem: $\iint_\Omega \left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \right) dx\, dy = \oint_{\delta\Omega} \phi\, dy - \oint_{\delta\Omega} \psi\, dx$

For instance, take

$$\phi = \eta \frac{\partial f}{\partial z_x} \quad \text{and} \quad \psi = \eta \frac{\partial f}{\partial z_y}$$

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= \eta_x \frac{\partial f}{\partial z_x} + \eta \frac{\partial}{\partial x} \frac{\partial f}{\partial z_x} \\ \frac{\partial \psi}{\partial y} &= \eta_y \frac{\partial f}{\partial z_y} + \eta \frac{\partial}{\partial y} \frac{\partial f}{\partial z_y} \end{aligned}$$

Green's theorem in use

Green's theorem: $\iint_\Omega \left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \right) dx\, dy = \int_{\delta\Omega} \phi\, dy - \int_{\delta\Omega} \psi\, dx$

So

$$\begin{aligned} &\iint_\Omega \left(\eta_x \frac{\partial f}{\partial z_x} + \eta_y \frac{\partial f}{\partial z_y} + \eta \frac{\partial}{\partial x} \frac{\partial f}{\partial z_x} + \eta \frac{\partial}{\partial y} \frac{\partial f}{\partial z_y} \right) dx\, dy \\ &= \oint_{\delta\Omega} \eta \frac{\partial f}{\partial z_x} dy - \oint_{\delta\Omega} \eta \frac{\partial f}{\partial z_y} dx \end{aligned}$$

Notice that $\eta(x,y) = 0$ for all $(x,y) \in \delta\Omega$, and so the right hand side integrals are both zero.

Given the RHS of the equation was zero, we can rearrange to get

$$\iint_\Omega \left(\eta_x \frac{\partial f}{\partial z_x} + \eta_y \frac{\partial f}{\partial z_y} \right) dx\, dy = - \iint_\Omega \eta \left[\frac{\partial}{\partial x} \frac{\partial f}{\partial z_x} + \frac{\partial}{\partial y} \frac{\partial f}{\partial z_y} \right] dx\, dy$$

With the result that the First Variation can be written

$$\delta F(\eta, z) = \iint_\Omega \eta \left[\frac{\partial f}{\partial z} - \frac{\partial}{\partial x} \frac{\partial f}{\partial z_x} - \frac{\partial}{\partial y} \frac{\partial f}{\partial z_y} \right] dx\, dy$$

This step is the analogy of integration by parts in the derivation of the standard Euler-Lagrange equation.

Euler-Lagrange equation

Given that $F(\eta, z) = 0$ for all allowable η, Lemma 2.2.2 (see last page) can be extended directly to the 2D case, with the result that

$$\frac{\partial f}{\partial z} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z_x} - \frac{\partial}{\partial y}\frac{\partial f}{\partial z_y} = 0$$

This is also called the Euler-Lagrange equation.

The general case of the Euler-Lagrange equations with 2 independent variables (and the boundary conditions) produces a Dirichlet boundary value problem.
these can be very hard to solve.

Simple example

Let Ω be the disk defined by $x^2 + y^2 < 1$, and the functional of interest be

$$F\{z\} = \iint_\Omega 1 + \frac{1}{2}z_x^2 + \frac{1}{2}z_y^2 \, dx\, dy$$

with boundary conditions

$$z_0(x,y) = 2x^2 - 1$$

for all (x,y) such that $x^2 + y^2 = 1$.

Simple example: solution

The Euler-Lagrange equation is

$$\frac{\partial f}{\partial z} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z_x} - \frac{\partial}{\partial y}\frac{\partial f}{\partial z_y} = 0$$

Note that in this example, f has no explicit dependence on x, y or z, and so we get

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

This equation is called Laplace's equation.

Consider the function $z = x^2 - y^2$. This satisfies Laplace's equation, and on the boundary $y^2 = 1 - x^2$, so $z = 2x^2 - 1$, which satisfies our boundary condition.

Example: vibrating string

- Imagine a taut string
  - flexible
  - uniform mass
  - small deflections

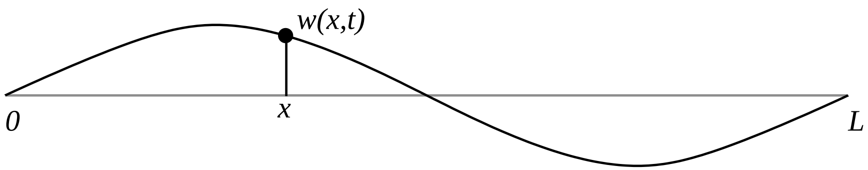
- Equilibrium solution
  - the string sits in a straight line
  - consider small perturbations

Example: vibrating string

Model:

- length of string is L
- position along the string is $x \in [0, L]$
- constant tension τ
- points on string move up/down perpendicular to x-axis
- displacement at x at time t is $w(x,t) \ll L$
- no friction or other damping
- only force occurs to stretch string
- constant density σ along the string's length

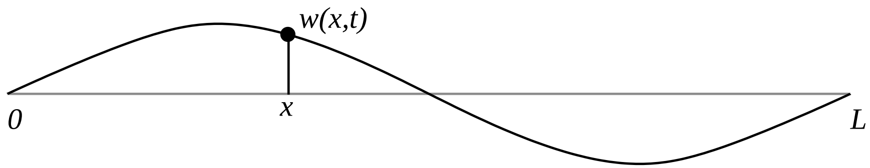
Example: vibrating string



- end points are fixed so $w(0,t) = w(L,t) = 0$
- velocity of particle is $w_t = \frac{\partial w}{\partial t}$
  - kinetic energy of string

$$T = \frac{\sigma}{2} \int_0^L w_t^2 \, dx$$

Example: vibrating string



- slope of string $\frac{\partial w}{\partial x}$
  - potential energy of the string depends on how much it is stretch from its original length L
    - length at time t is given by

$$J(t) = \int_0^L \sqrt{1 + w_x^2} \, dx$$

Example: vibrating string

- potential is $V = \tau(J - L)$, so

$$V(t) = \tau \int_0^L \sqrt{1 + w_x^2} - 1 \, dx$$

- we assumed that w is small, so we can approximate

$$\sqrt{1 + w_x^2} \simeq 1 + \frac{1}{2} w_x^2$$

- so we use

$$V(t) = \frac{\tau}{2} \int_0^L w_x^2 \, dx$$

Example: vibrating string

The system is conservative so we apply the "principle of least action" (Hamilton's principle), which says the shape will be an extremum with respect to

$$F\{w\} = \int_{t_1}^{t_2} (T - V)\, dt = \frac{1}{2}\int_{t_1}^{t_2}\int_0^L \sigma w_t^2 - \tau w_x^2 \, dx\, dt$$

The Euler-Lagrange equation is

$$\frac{\partial f}{\partial w} - \frac{\partial}{\partial x}\frac{\partial f}{\partial w_x} - \frac{\partial}{\partial t}\frac{\partial f}{\partial w_t} = 0$$

which gives

$$\frac{\partial}{\partial x}\tau w_x = \frac{\partial}{\partial t}\sigma w_t$$

Example: vibrating string

$$\frac{\partial}{\partial x}\tau w_x = \frac{\partial}{\partial t}\sigma w_t$$

or

$$\frac{\partial^2 w}{\partial x^2} = \frac{\sigma}{\tau}\frac{\partial^2 w}{\partial t^2}$$

which is the classic wave equation, which you have no doubt seen solved in other contexts.

Example: Plateau's problem

We want to find the surface with minimal area stretched between a boundary.

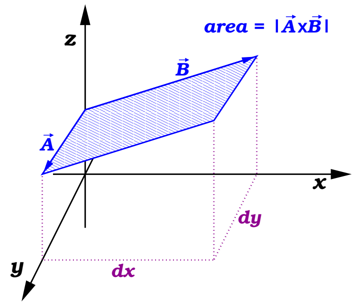
- this is what a soap film does
- architecture influenced by minimal surfaces
  - architect Frei Otto
  - Munich Olympic Stadium

Surface area minimization

The functional of interest is the surface area

$$F\{z\} = \int_\Omega dS$$

As before, we can't compute this integral, so we must convert it to a convenient form:



$$\begin{aligned} \mathbf{A} &= (0, dy, z_y\, dy) \\ \mathbf{B} &= (dx, 0, z_x\, dx) \\ \mathbf{A}\times\mathbf{B} &= (z_x\, dx\, dy, z_y\, dx\, dy, -dx\, dy) \end{aligned}$$

$$\begin{aligned} dS &= |\mathbf{A}\times\mathbf{B}| = \sqrt{(z_x\, dx\, dy)^2 + (z_y\, dx\, dy)^2 + (-dx\, dy)^2} \\ &= dx\, dy\sqrt{1 + z_x^2 + z_y^2} \end{aligned}$$

Surface area minimization

So we may rewrite the functional as

$$F\{z\} = \iint_\Omega \sqrt{1+z_x^2+z_y^2}\, dx\, dy$$

The Euler-Lagrange equation is

$$\frac{\partial f}{\partial z} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z_x} - \frac{\partial}{\partial y}\frac{\partial f}{\partial z_y} = 0$$

Which in this context is

$$-\frac{\partial}{\partial x}\left[\frac{z_x}{\sqrt{1+z_x^2+z_y^2}}\right] - \frac{\partial}{\partial y}\left[\frac{z_y}{\sqrt{1+z_x^2+z_y^2}}\right] = 0$$

Surface area minimization

Continuing the derivation

$$\begin{aligned}
\frac{\partial}{\partial x}\left[\frac{z_x}{\sqrt{1+z_x^2+z_y^2}}\right] &= \frac{z_{xx}}{\sqrt{1+z_x^2+z_y^2}} - \frac{z_x(z_x z_{xx} + z_y z_{yx})}{(1+z_x^2+z_y^2)^{3/2}} \\
&= \frac{z_{xx}(1+z_x^2+z_y^2) - z_x(z_x z_{xx} + z_y z_{yx})}{(1+z_x^2+z_y^2)^{3/2}} \\
&= \frac{z_{xx}(1+z_y^2) - z_x z_y z_{yx}}{(1+z_x^2+z_y^2)^{3/2}} \\
\frac{\partial}{\partial y}\left[\frac{z_y}{\sqrt{1+z_x^2+z_y^2}}\right] &= \frac{z_{yy}(1+z_x^2) - z_x z_y z_{yx}}{(1+z_x^2+z_y^2)^{3/2}}
\end{aligned}$$

Surface area minimization

Add the two terms above to get the E-L equation

$$2\mathcal{C} = \frac{z_{xx}(1+z_y^2) - 2z_x z_y z_{yx} + z_{yy}(1+z_x^2)}{(1+z_x^2+z_y^2)^{3/2}} = 0$$

where we call $\mathcal{C}$ the mean curvature (which is 0 on the extremals).

We multiply both sides of the E-L equation by the denominator to get

$$z_{xx}(1+z_y^2) - 2z_x z_y z_{yx} + z_{yy}(1+z_x^2) = 0$$

This is a hard PDE in general.

Approximate solutions

If the surfaces are almost planes (e.g. if z is small), then we can take squared derivate terms like z_x^2, z_y^2 and $z_x z_y$ to be zero. In this case the general equation

$$z_{xx}(1+z_y^2) - 2z_x z_y z_{yx} + z_{yy}(1+z_x^2) = 0$$

simplifies to give us

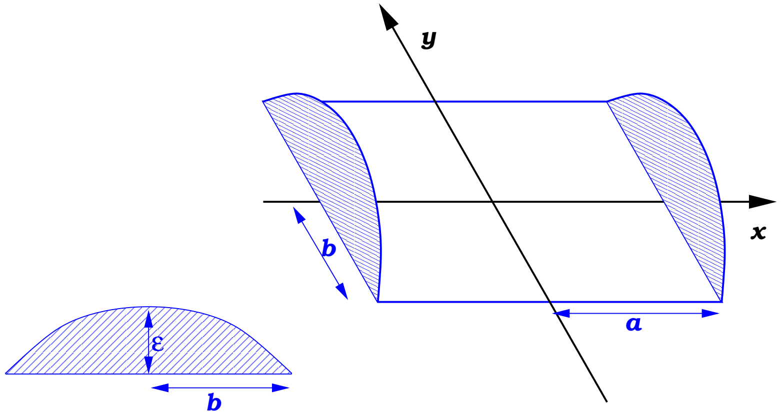
$$z_{xx} + z_{yy} = 0$$

the Laplace equation again. We know from the previous example that this is equivalent to approximating

$$f(x,y,z,z_x,z_y) = \sqrt{1+z_x^2+z_y^2} \simeq 1 + \frac{1}{2}z_x^2 + \frac{1}{2}z_y^2$$

Example

Design a surfaces of minimum surface area over a stadium with small curved walls, of shape $z = \varepsilon \cos\left(\frac{\pi}{2}\frac{y}{b}\right)$, located at $x = \pm a$, and with no end walls at $y = \pm b$.



Example

Use the approximation, so we wish to solve

$$\begin{aligned} z_{xx} + z_{yy} &= 0 \\ z(\pm a, y) &= \varepsilon \cos\left(\frac{\pi}{2}\frac{y}{b}\right) \\ z(x, \pm b) &= 0 \end{aligned}$$

Assume a solution with separation of variables, e.g. $z(x,y) = X(x)Y(y)$, then the DE implies that

$$z \propto \begin{matrix}\cosh \\ \sinh\end{matrix}(\lambda y) \times \begin{matrix}\cos \\ \sin\end{matrix}(\lambda y)$$

Choose cos with $\lambda = \frac{\pi}{2b}$ to match the boundary conditions, and choose cosh because we expect the solution to be even.

Example: solution

So the solution is

$$z(x,y) = A\cos\left(\frac{\pi y}{2b}\right)\cosh\left(\frac{\pi x}{2b}\right)$$

Determine A using the end-points, e.g.

$$\varepsilon \cos\left(\frac{\pi y}{2b}\right) = A\cos\left(\frac{\pi y}{2b}\right)\cosh\left(\frac{\pi a}{2b}\right)$$

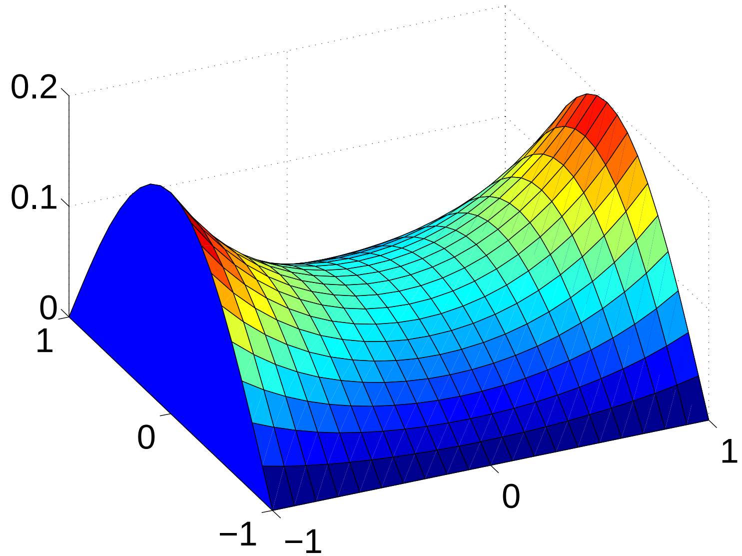
So

$$A = \varepsilon / \cosh\left(\frac{\pi a}{2b}\right)$$

and

$$z(x,y) = \varepsilon \cos\left(\frac{\pi y}{2b}\right)\cosh\left(\frac{\pi x}{2b}\right) / \cosh\left(\frac{\pi a}{2b}\right)$$

Example: solution



Example: solution

In fact, once we realize it will have a cosine cross-section, we know that the “area” of the curve for any given x will be proportional to the height, so we are in fact solving a problem that looks a lot like that of the catenary. So we should be surprised to see that the result has the same cosh function.

But this is hard...

Solving the PDE form of the EL equations can be very hard. What can we do to make it easier? Surely computers can help?

Plateau’s laws

A little bit extra:

- Soap films are made of entire smooth surfaces
- The average curvature of a portion of a soap film is always constant on any point on the same piece of soap film
- Soap films always meet in threes, and they do so at an angle of $cos^{-1}(-1/2) = 120$ degrees forming an edge called a Plateau Border.
- Plateau Borders meet in fours at an angle of $\cos^{-1}(-1/3) \simeq 109.47$ degrees (the tetrahedral angle) to form a vertex.