Variational Methods & Optimal Control

lecture 16

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Non-integral constraints

It is relatively easy to adapt the Lagrange multiplier technique to the case with non-integral constraints.

- **Holonomic constraints**[a] are of the form

$$g(t, \mathbf{q}) = 0$$

- **Non-Holonomic constraints** are of the form

$$g(t, \mathbf{q}, \dot{\mathbf{q}}) = 0$$

The former is simpler, and we consider this first.

[a] Holonomic comes from the greek "holos", for "whole". In this context it refers to integrability of the constraint. Notice that non-holonomic constraints are really DEs

Holonomic Constraints

Constraints of the form $g(x,y) = 0$, or $g(t,\mathbf{q}) = 0$, which don't involve derivatives of $y(x)$ or $\mathbf{q}$ can also be handled using a Lagrange multiplier technique, but we have to introduce a Lagrange multiplier function $\lambda(x)$, not just a single value λ. Effectively we introduce one Lagrange multiplier at each point where the constraint is enforced.

Holonomic constraints

Consider the problem of finding extremals of

$$F\{y\} = \int_{x_0}^{x_1} f(x,y,y')\,dx$$

subject to the constraint

$$g(x,y) = 0$$

In this case we introduce a function $\lambda(x)$ (also called a Lagrange multiplier), and look for extremals of

$$H\{y,\lambda\} = F\{y\} + \int_{x_0}^{x_1} \lambda(x) g(x,y) dx$$

Why does it work

Go back to the finite approximation Consider Euler's finite difference method on a uniform grid for approximation of the functional

$$F\{y\} = \int_a^b f(x,y,y')\,dx \simeq \sum_{i=1}^{n} f\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right)\Delta x = \bar{F}(\mathbf{y})$$

The constraint applies a condition on each (x_i, y_i), so in the approximation there are n constraints,

$$g(x_i, y_i) = 0 \text{ for } i = 1, \ldots, n$$

Why does it work

There are n constraints,

$$g(x_i, y_i) = 0 \text{ for } i = 1, \ldots, n$$

For optimization problems with n constraints, we introduce n Lagrange multipliers, and maximize

$$H(\mathbf{y}) = F(\mathbf{y}) + \sum_{k=1}^{n} \lambda_k g(x_k, y_k)$$

In the limit as $n \to \infty$

$$\Delta x \sum_{k=1}^{n} \lambda_k g(x_k, y_k) \to \int_a^b \lambda(x) g(x, y)\, dx$$

and hence the choice of $H\{y, \lambda\} = F\{y\} + \int_a^b \lambda(x) g(x, y) dx$.

Holonomic constraints

$$
\begin{aligned}
H\{y, \lambda\} &= F\{y\} + \int_{x_0}^{x_1} \lambda(x) g(x, y)\, dx \\
&= \int_{x_0}^{x_1} f(x, y, y') + \lambda(x) g(x, y)\, dx
\end{aligned}
$$

So we can apply the Euler-Lagrange equations to

$$
h(x, y, y', \lambda) = f(x, y, y') + \lambda(x) g(x, y)
$$

To get the Euler-Lagrange equations

$$
\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} - \lambda(x)\frac{\partial g}{\partial y} = 0
$$

Multiple dependent variables

With multiple dependent variables holonomic constraints are of the form

$$g(t, \mathbf{q}) = 0$$

and they don't involve derivatives.

Example: find geodesics on a cylinder, e.g. minimize

$$F\{x, y, z\} = \int_{t_0}^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$$

subject to $x^2 + y^2 - r^2 = 0$, the equation of a right circular cylinder with radius r.

Multiple dependent variables

$$H\{\mathbf{q},\lambda\} = F\{\mathbf{q}\} + \int_{x_0}^{x_1} \lambda(t) g(t,\mathbf{q})\, dx$$

So we can apply the Euler-Lagrange equations to

$$h(t,q,\dot{q},\lambda) = f(t,q,\dot{q}) + \lambda(t) g(t,q)$$

To get the Euler-Lagrange equations

$$\frac{d}{dx}\frac{\partial f}{\partial \dot{q}_k} - \frac{\partial f}{\partial q_k} - \lambda(t)\frac{\partial g}{\partial q_k} = 0$$

for all k.

General geodesic problem

General geodesic problem can be written as minimize

$$F\{x,y,z\} = \int_{t_0}^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$$

subject to

$$g(x,y,z) = 0$$

where $g(x,y,z) = 0$ is the equation describing the surface of interest.

We instead minimize

$$H\{x,y,z,\lambda\} = \int_{t_0}^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} + \lambda(t) g(x,y,z)\, dt$$

General geodesic problem

Given this formulation of the geodesic problem, the E-L equations become

$$
\begin{aligned}
\frac{d}{dt}\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} - \lambda(t)\frac{\partial g}{\partial x} &= 0 \\
\frac{d}{dt}\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} - \lambda(t)\frac{\partial g}{\partial y} &= 0 \\
\frac{d}{dt}\frac{\dot{z}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} - \lambda(t)\frac{\partial g}{\partial z} &= 0
\end{aligned}
$$

which may be easier to solve in some cases.

Example: Geodesics on the sphere

Find the geodesics on the sphere: e.g. we wish to find a parametric curve $(x(t), y(t), z(t))$ to minimize distance

$$F\{x,y,z\} = \int_{t_0}^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$$

subject to being on the surface of a sphere

$$x^2 + y^2 + z^2 = a^2$$

We get

$$h(t,x,y,z,\dot{x},\dot{y},\dot{z}) = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} + \lambda(t)(x^2 + y^2 + z^2)$$

and there are three dependent variables (x,y,z)

Example: Geodesics on the sphere

$$
\begin{aligned}
\frac{\partial h}{\partial x} &= 2\lambda x & \qquad \frac{\partial h}{\partial \dot{x}} &= \frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} \\
\frac{\partial h}{\partial y} &= 2\lambda y & \qquad \frac{\partial h}{\partial \dot{y}} &= \frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} \\
\frac{\partial h}{\partial z} &= 2\lambda z & \qquad \frac{\partial h}{\partial \dot{z}} &= \frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}
\end{aligned}
$$

Example: Geodesics on the sphere

There are 3 dependent variables (x, y, z), and so 3 E-L equations, e.g.

$$
\begin{aligned}
2\lambda x &= \frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\right) \\
&= \frac{\ddot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} - \frac{\dot{x}(\dot{x}\ddot{x}+\dot{y}\ddot{y}+\dot{z}\ddot{z})}{(\dot{x}^2+\dot{y}^2+\dot{z}^2)^{3/2}}
\end{aligned}
$$

Due to symmetry, the equation

$$
2\lambda u = \frac{\ddot{u}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} - \frac{\dot{u}(\dot{x}\ddot{x}+\dot{y}\ddot{y}+\dot{z}\ddot{z})}{(\dot{x}^2+\dot{y}^2+\dot{z}^2)^{3/2}}
$$

holds for $u = x, y$ and z.

Example: Geodesics on the sphere

Now

$$2\lambda u = \ddot{u}\frac{1}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} - \dot{u}\frac{(\dot{x}\ddot{x}+\dot{y}\ddot{y}+\dot{z}\ddot{z})}{(\dot{x}^2+\dot{y}^2+\dot{z}^2)^{3/2}}$$

is a second order linear DE in u, and so it has only 2 linearly independent solutions, but the DE holds for
$u = x, y$ and z

Therefore x, y, and z are linearly dependent, and so we can write them as

$$Ax + By + Cz = 0$$

but this is the equation of a plane through the origin. Once again we have shown that geodesics on the sphere are great circles

Misc

- Note, sometimes a constraint involving derivatives may be integrated to get a holonomic constraint, so we refer to these constraints as integrable.
- In general, though, we will also need to deal with constraints involving derivatives as these may describe an entire systems behaviour, and be very difficult to integrate out of the problem.
    - e.g., when we want to describe a “controlled” system

Non-Holonomic Constraints

Constraints of the form $g(x, y, y') = 0$, or $g(t, \mathbf{q}, \dot{\mathbf{q}}) = 0$, which involve derivatives. They are effectively additional DEs which we need to solve, but we can once again use Lagrange multipliers.

Non-holonomic constraints

Example non-holonomic constraints:

$$g(x,y,y') = 0 \qquad \text{or} \qquad g(t,\mathbf{q},\dot{\mathbf{q}}) = 0$$

Instances:

- $y = \dot{x}$
- $y'^2 = \log x$

Solution technique is just as for holonomic constraints, e.g.,

$$H\{y,\lambda\} = F\{y\} + \int_{x_0}^{x_1} \lambda(x) g(x,y,y')\, dx$$

and the argument for why it works is almost identical.

Example

Using such constraints to avoid higher derivatives

Imagine the functional

$$F\{y\} = \int_a^b f(x, y, y', y'')\, dx$$

we have already see that we can derive a new form of the E-L (Euler-Poisson) equations for this case, e.g.

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} = 0$$

but constraints give us an alternative approach to this problem.

Example

Introduce the new variable $z = y'$, and rewrite the functional as

$$F\{y\} = \int_a^b f(x, y, z, z')\, dx$$

Now there is more than one dependent variable, but no second order derivatives, however, we must also introduce the constraint that

$$z - y' = 0$$

and so we look for stationary curves of the functional

$$H\{y, z, \lambda\} = \int_a^b f(x, y, z, z') + \lambda(x)(z - y')\, dx$$

Example

The Euler-Lagrange equations for y and z are

$$\begin{aligned} \frac{d}{dx}\frac{\partial h}{\partial y'} - \frac{\partial h}{\partial y} &= 0 \\ \frac{d}{dx}\frac{\partial h}{\partial z'} - \frac{\partial h}{\partial z} &= 0 \end{aligned}$$

note that $h(x,y,z,z') = f(x,y,z,z') + \lambda(x)(z - y')$ so the E-L equations become

$$\begin{aligned} \frac{d}{dx}[-\lambda(x)] - \frac{\partial f}{\partial y} &= 0 \\ \frac{d}{dx}\frac{\partial f}{\partial z'} - \frac{\partial f}{\partial z} - \lambda(x) &= 0 \end{aligned}$$

Example

The first Euler-Lagrange equation can be rewritten

$$\frac{d\lambda}{dx} = -\frac{\partial f}{\partial y}$$

Differentiating the second E-L equation WRT x we get

$$\frac{d^2}{dx^2}\frac{\partial f}{\partial z'} - \frac{d}{dx}\frac{\partial f}{\partial z} - \frac{d\lambda}{dx} = 0$$

Note from above that $\lambda' = -f_y$ and that $z = y'$ and $z' = y''$ we get (as before) the Euler-Poisson equation:

$$\frac{d^2}{dx^2}\frac{\partial f}{\partial y''} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{\partial f}{\partial y} = 0$$

Some intuition

- Earlier we derived the Euler-Lagrange equations assuming treating y and y' as if they were independent variables.
  - In reality they are related along the extremal
  - Lets get some motivation for this
- Start by taking a new variable $u(x) = y'(x)$, and put this into our optimization problem

$$H\{y,u,\lambda\} = \int_a^b f(x,y,u) + \lambda(x)(u - y')\,dx$$

  - we can use same trick as in previous slides to get the Euler-Lagrange equations

Newton's aerodynamical problem

Find extremal of "air resistance"

$$F\{y\} = \int_0^R \frac{x}{1+y'^2}\,dx,$$

subject to $y(0) = L$ and $y(R) = 0$ and $y' \leq 0$ and $y'' \geq 0$
The Euler-Lagrange equations are

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = \frac{d}{dx}\frac{2xy'}{(1+y'^2)^2} = 0$$

Rearranging we get

$$2xy' = c(1+y'^2)^2$$

which isn't much fun to solve directly.

Newton's aerodynamical problem

Alternative: define a new variable u, and constrain it

$$u = -y'$$

Add Lagrange multiplier $\lambda(x)$ to the functional

$$H\{y,u,\lambda\} = \int_0^R \frac{x}{1+u^2} + \lambda(y'+u)\,dx,$$

Now solve as you would for a problem with three dependent variables (y,u,λ) of x.

- We expect three Euler-Lagrange equations
- One equation in each dependent variable
  - but we already know the λ equation

Newton's aerodynamical problem

Euler-Lagrange equations

$$\begin{aligned}
\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} &= 0 \\
\frac{d}{dx}\frac{\partial f}{\partial u'} - \frac{\partial f}{\partial u} &= 0 \\
\frac{d}{dx}\frac{\partial f}{\partial \lambda'} - \frac{\partial f}{\partial \lambda} &= 0
\end{aligned}$$

give the DEs

$$\begin{aligned}
\lambda &= const \\
-\frac{2xu}{(1+u^2)^2} - \lambda &= 0 \\
y' + u &= 0
\end{aligned}$$

Newton's aerodynamical problem

$$\begin{aligned} \lambda &= const \\ -\frac{2xu}{(1+u^2)^2} - \lambda &= 0 \\ y' + u &= 0 \end{aligned}$$

If $\lambda = 0$, then for $x > 0$ we get $u = 0$, and hence $y = const$.
If $\lambda \neq 0$ then the second equation implies

$$x(u) = \frac{c}{u}(1+u^2)^2 = c\left(\frac{1}{u} + 2u + u^3\right).$$

for c constant.

Newton’s aerodynamical problem

From the last equation (which we insisted on at the start), we get

$$\frac{dy}{dx} = -u$$

Now note that from the chain rule

$$\begin{aligned}
\frac{dy}{du} &= \frac{dy}{dx}\frac{dx}{du} = -u\frac{dx}{du} \\
&= c\left(-\frac{1}{u} + 2u + 3u^3\right)
\end{aligned}$$

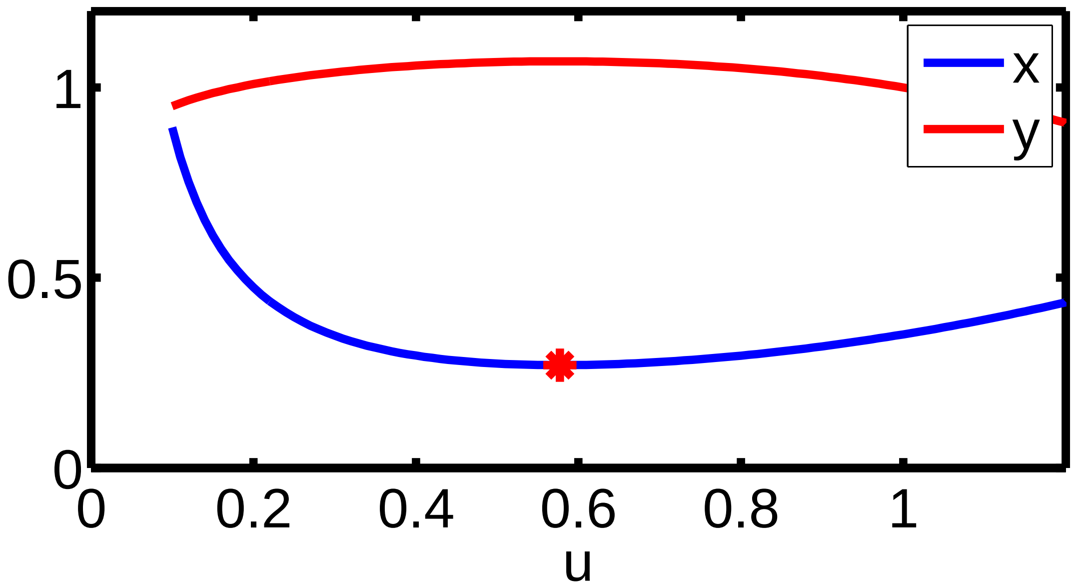
which we can integrate with respect to u to get

$$y(u) = const - c\left(-\ln u + u^2 + \frac{3}{4}u^4\right)$$

Newton's aerodynamical problem

Some notes about the solution

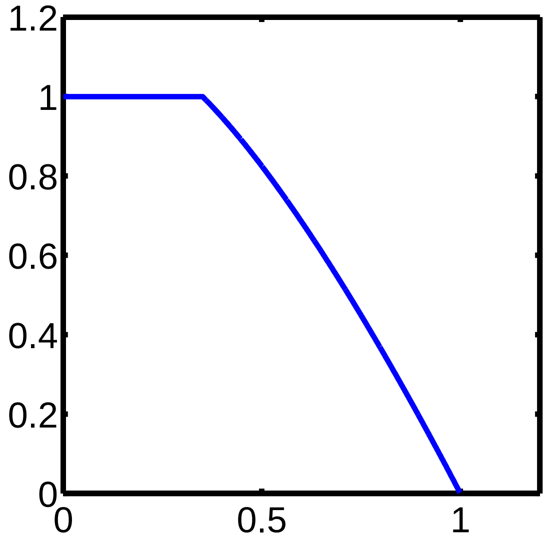
- $x(u) = \frac{c}{u}(1+u^2)^2 > 0$ for all u (because $u = -y' > 0$)



- hence the part of the curve near $x = 0$ must have $y = L$
- but $y(x) = L$ for all $x \in [0, R]$ can't be the minimum because we know better profiles (e.g. a cone).

Newton’s aerodynamical problem

So we know the solution must look like something like



A simple example is the **frustum of a cone**

- the part of a cone between two parallel planes
- but we can do better by making the sloped part follow E-L equations
- still need to work out where the “corner” goes

Newton's aerodynamical problem

Equations: for curved part

$$
\begin{aligned}
x(u) &= c\left(\frac{1}{u} + 2u + u^3\right) \\
y(u) &= const - c\left(-\ln u + u^2 + \frac{3}{4}u^4\right)
\end{aligned}
$$

End points conditions:

$$
\begin{aligned}
y(u_1) &= L \\
y(u_2) &= 0 \\
x(u_1) &= x_1 \\
x(u_2) &= R
\end{aligned}
$$

but we don't know x_1, u_1 or u_2.

Newton's aerodynamical problem

$$y(u) = const - c\left(-\ln u + u^2 + \frac{3}{4}u^4\right)$$

At u_1 we have $y(u_1) = L$. For convenience we write

$$y(u) = L - c\left(-A - \ln u + u^2 + \frac{3}{4}u^4\right)$$

so at u_1 we get

$$\begin{aligned}
L &= L - c\left(-\ln u_1 - A + u_1^2 + \frac{3}{4}u_1^4\right)\\
0 &= -c\left(-\ln u_1 - A + u_1^2 + \frac{3}{4}u_1^4\right)\\
A &= -\ln u_1 + u_1^2 + \frac{3}{4}u_1^4
\end{aligned}$$

Newton’s aerodynamical problem

$$
\begin{aligned}
y(u) &= L - c\left(-A - \ln u + u^2 + \frac{3}{4}u^4\right) \\
x(u) &= \frac{c}{u}(1+u^2)^2
\end{aligned}
$$

Now at u_2 we have $x(u_2) = R$ and $y(u_2) = 0$ so

$$
\begin{aligned}
L &= c\left(-A - \ln u_2 + u_2^2 + \frac{3}{4}u_2^4\right) \\
R &= \frac{c}{u_2}(1+u_2^2)^2
\end{aligned}
$$

divide the first equation by the second and we get ...

Newton's aerodynamical problem

$$\frac{L}{R} = u_2 \left(-A - \ln u_2 + u_2^2 + \frac{3}{4} u_2^4 \right) (1 + u_2^2)^{-2}$$

The function on the RHS is increasing so we can solve this equation (numerically (e.g., using matlab's `fsolve`), and we obtain a value for u_2.
We can find c using $x(u_2) = R$

$$\begin{aligned} R &= \frac{c}{u_2}(1+u_2^2)^2 \\ c &= \frac{u_2 R}{(1+u_2^2)^2} \end{aligned}$$

All we need to know now is u_1, which gives us A and $x(u_1)$, which gives us u_2, which gives us c.

Newton's aerodynamical problem

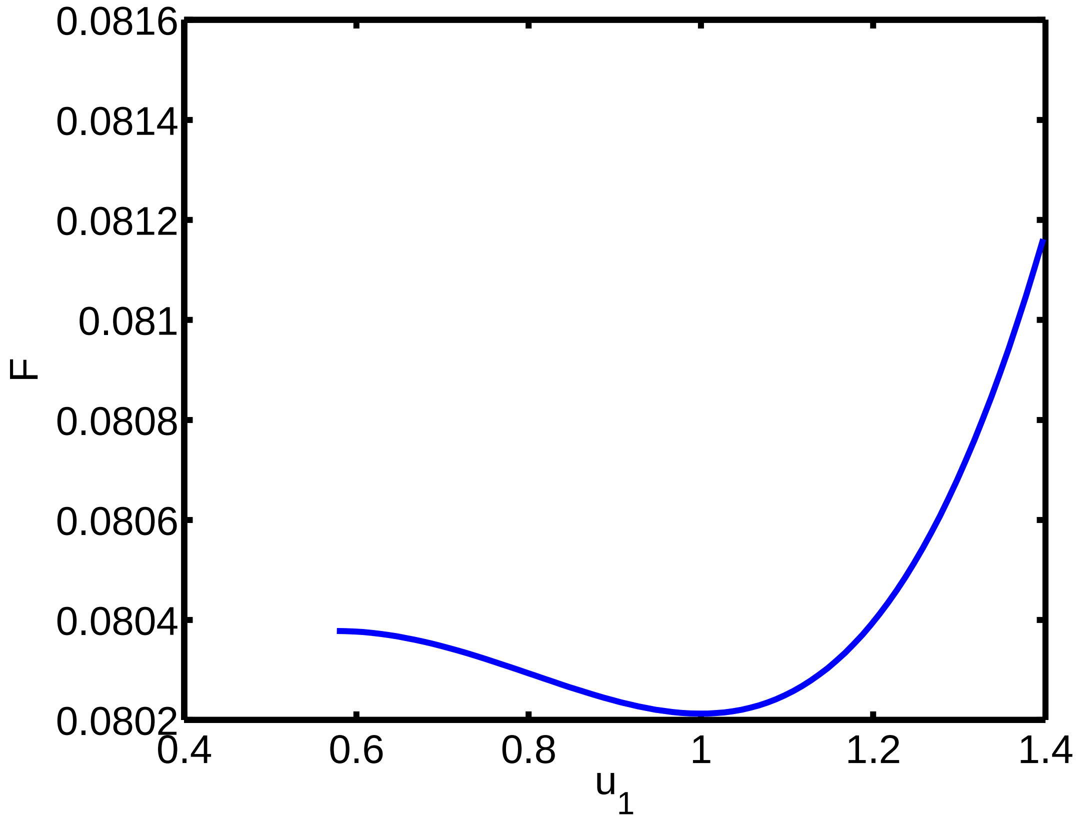
Take $x(u_1) = x_1$

$$
\begin{aligned}
F\{y\} &= \int_0^{x_1} x\,dx + \int_{x_1}^{R} \frac{x}{1+y'^2}\,dx \\
&= \frac{x_1^2}{2} + \int_{u_1}^{u_2} \frac{c(1+u^2)^2/u}{1+u^2}\,\frac{dx}{du}\,du \\
&= \frac{x_1^2}{2} + c^2 \int_{u_1}^{u_2} \frac{(1+u^2)^2(3u^2-1)}{u^3}\,du \\
&= \frac{x_1^2}{2} + c^2 \left[\frac{3u^4}{4} + \frac{5u^2}{2} + \ln(u) + \frac{1}{2u^2} \right]_{u_1}^{u_2}
\end{aligned}
$$

and note that c and u_2 are effectively functions of u_1.

Newton’s aerodynamical problem

Numerical evaluation of the integral F for different values of if u_1



Minimum occurs for $u_1 = 1$, we’ll prove this later on.

Intro to Optimal Control

One way we see non-holonomic constraints is when we consider control problems. In these we seek to control a system described by a DE (the constraint) subject to some input which we can control (optimize).

Systems

system = machine + controller

e.g. vehicle

- **machine:** engine, body, seating
- **control:** accelerator, brakes, steering (driver)

Problems:

- **Control problems:** how do we set, say the steering and acceleration of a car to get it from point A to point B.
- **Optimal Control problems:** same as above, but do it in minimum time, or using minimum fuel.

Solution Philosophy

Solve however you can

- often easier approach that CoV
  - systems of DEs just need to be solved
  - a lot is about whether a control exists!
  - e.g. see "Optimal Control: an Introduction to the Theory with Applications", Leslie M. Hocking, Clarendon Press, Oxford, 1991.
- on the other hand, we have a powerful set of tools now, so we shall use them here
  - all it takes is a shift in perspective
  - then all of the CoV work from earlier is applicable

CoV for Optimal Control

Optimal control is just a switch in our perspective:

- previous problems, mainly concerned with modeling and analysis of physical (often mechanical systems), e.g. catenary
  - take a system, and find an extremal which minimizes, say potential energy, and this describes the system
- now we can set part of the systems (e.g. force) to create a particular curve which minimizes some quantity
  - e.g. set force to minimize fuel usage of a rocket (changing orbits)

Formulation of control problems

We break a control problem into two parts

- **The system state:** $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))^t$
  The system state describes the system (e.g. position and velocity of the car in car parking example)
- **The control:** $\mathbf{u}(t) = (u_1(t), \ldots, u_m(t))^t$
  We apply the control to the system (e.g. force applied to the car).

The evolution of the system is governed by a DE

$$\dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}, \mathbf{u})$$

In a control problem we control the system to get it to a particular state $\mathbf{x}(t_1)$ at time t_1, given initial state $\mathbf{x}(t_0)$.

Optimal control problems

In an optimal control problem we have

$$\dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}, \mathbf{u})$$

and once again we wish to get to state $\mathbf{x}(t_1)$ given initial state $\mathbf{x}(t_0)$, but now we wish to do so while minimizing a functional

$$F\{\mathbf{u}\} = \int_{t_0}^{t_1} f(t, \mathbf{x}, \mathbf{u})\, dt$$

That is, we wish to choose a function $\mathbf{u}(t)$ which minimizes the functional $F\{\mathbf{u}\}$, while satisfying the end-point conditions $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{x}(t_1) = \mathbf{x}_1$, and the non-holonomic constraint $\dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}, \mathbf{u})$.

Example: stimulated plant growth

Plant growth problem:

- market gardener wants to plants to grow to a fixed height 2 within a fixed window of time $[0,1]$
- can supplement natural growth with lights with "brightness" $u(t)$
- growth rate dictates

$$\dot{x} = 1 + u$$

- cost of lights

$$F\{u\} = \int_0^1 \frac{1}{2} u(t)^2 \, dt$$

Plant growth problem statement

Minimize

$$F\{u\} = \int_0^1 \frac{1}{2} u^2 \, dt$$

Subject to $x(0) = 0$, and $x(1) = 2$ and

$$\dot{x} = 1 + u$$

- we effectively have two dependent variables x and u
- though we can only control one of these

Plant growth: Lagrange multiplier

We can include the non-holonomic constraint into the problem via a Lagrange multiplier, e.g., we seek to minimize

$$
\begin{aligned}
H\{u, x, \lambda\} &= \int_0^1 \frac{1}{2}u^2 + \lambda(t)\left[\dot{x} - 1 - u\right] dt \\
&= \int_0^1 h(x, u, \dot{x}, \lambda)\, dt
\end{aligned}
$$

We might think of λ as a third variable, but the E-L equations in λ will just give us the constraint $\dot{x} = 1 + u$ back again.

Plant growth: E-L equations

2 dependent variables, so E-L equations

$$\begin{aligned} \frac{\partial h}{\partial u} - \frac{d}{dt}\frac{\partial h}{\partial \dot{u}} &= 0 \\ \frac{\partial h}{\partial x} - \frac{d}{dt}\frac{\partial h}{\partial \dot{x}} &= 0 \end{aligned}$$

These are

$$\begin{aligned} u - \lambda &= 0 \\ \dot{\lambda} &= 0 \end{aligned}$$

Simplifying we see the solution is

$$u = const$$

Plant growth solution

Going back to the DE $\dot{x} = 1 + u$, and taking $u = c$ we get

$$x = (c+1)t + k$$

The end-point constraints require that $x(0) = 0$ and $x(1) = 2$ so

$$x = 2t$$

Clearly $u = 1$ is the optimal control.
We can also derive the optimal cost

$$F\{u\} = \int_0^1 \frac{1}{2} u(t)^2 \, dt = \frac{1}{2}$$

Optimal Control

We will consider optimal control much more thoroughly later in this course. There are many approaches one can adopt to such problems, and we shall come back to this problem in particular, later in the course. First we need to know some more CoV, especially how to deal with

- free end points
  - say there isn't a fixed time window
  - perhaps the final state isn't pre-determined
- costs other than integrals
  - e.g., costs associated with end states