Variational Methods & Optimal Control

lecture 21

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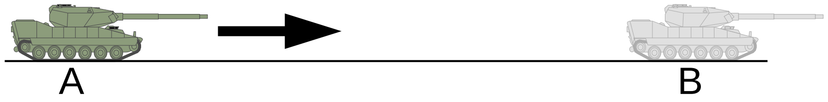
Inequality Constraints and Optimal Control

Earlier we didn't consider inequalities as constraints, but these are needed particularly in control. For instance, often there is a maximum force we can apply to an object. The resulting extremals either (i) satisfy the E-L equations, or (ii) lie along the edge of the constraint. We also get boundary conditions between these two types of regions.

Example: parking a car

Classic problem: from Craggs, p.55

> We want to drive a car/tank from point *A* to point *B* as quickly as possible, and at point *B* the car should be stationary.



Example

Parking a car seems like a trivial problem:

- in fact this problem appears in other contexts, e.g.
  - automatic positioning of components on a circuit board
  - has to be done frequently (so has to be fast)
  - speed limited by robot, and how delicate the components are
- shortest-time problems are a case of a more general type of problem as well.
- further, this type of controller appears often
  - we can make some general statements about when a bang-bang controller is a good idea

Example

tp://www.expo21xx.com/automation77/news/2085_robot_mitsubishi/news_default.htm

Example: parking a car

We want to drive a car/tank from point A to point B as quickly as possible, and at point B the car should be stationary.

Newton's law

$$\text{force} = u = m\ddot{x}$$

Choose force u that minimizes the time subject to $\dot{x} = 0$ at $t = 0$ and $t = T$, where T is not specified, but rather given by

$$T\{u\} = \int_A^B dt$$

and it is this functional we wish to minimize.

Example: parking a car

As before, note $\dot{x}(t) = dx/dt$ is the car's velocity, so we can write

$$T\{x\} = \int_A^B dt = \int_{x_A}^{x_B} \frac{1}{\dot{x}}\, dx$$

We wish to maximize this extremal, subject to the DE constraint that

$$\ddot{x} = \frac{u(t)}{m}$$

where $u(t)$ is the control (force) that we exert, and also subject to

$$\dot{x}(0) = \dot{x}(T) = 0$$

i.e., the car is stationary at the start and finish.

Example: parking a car

Take $y = \dot{x}$, and we can rewrite the problem as minimize

$$T\{y\} = \int_A^B dt = \int_{x_A}^{x_B} \frac{1}{y}\, dx$$

We wish to minimize this extremal, subject to the DE constraint that

$$\dot{y} = \frac{u(t)}{m}$$

where $u(t)$ is the control (force) that we exert, and also subject to

$$y(x_A) = y(x_B) = 0$$

Example: parking a car

Including the non-holonomic constraint into the problem using a Lagrange multiplier we get

$$H\{y,u\} = \int_{x_A}^{x_B} \frac{1}{y} + \lambda \left(\dot{y} - \frac{u(t)}{m} \right) dx$$

subject to

$$y(x_A) = y(x_B) = 0$$

The E-L equations are

$$\begin{aligned} \frac{d}{dt}\frac{\partial h}{\partial \dot{y}} - \frac{\partial h}{\partial y} &= 0 \\ \frac{d}{dt}\frac{\partial h}{\partial \dot{u}} - \frac{\partial h}{\partial u} &= 0 \end{aligned}$$

Example: parking a car

$$\begin{aligned} \frac{d}{dt}\lambda + \frac{1}{y^2} &= 0 \\ \frac{\lambda}{m} &= 0 \end{aligned}$$

From the second equation $\lambda = 0$, and so we see that
So the only viable solutions are $y = \pm\infty$

Example: parking a car

E-L solutions:

- solutions are $y = \pm\infty$
- this requires $u = \pm\infty$ at some points in time
- but in reality we can't exert infinite force
  - i.e., force is bounded

$$|u| \leq u_{\max}$$

- need to consider optimizing functionals with inequality constraints.
  - similar (in some respects) to min/max functions with inequality constraints
  - min/max is in the interior, or on the boundary

Inequality constraints

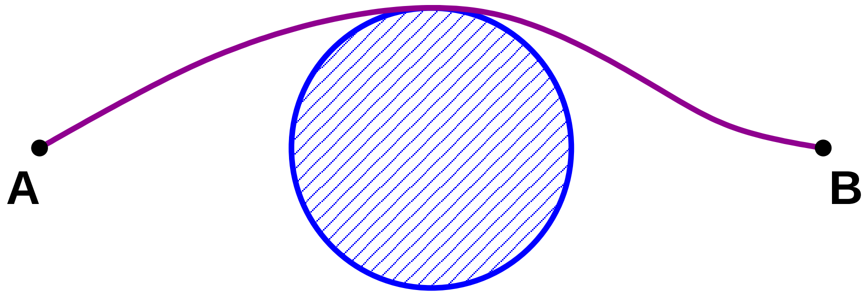
We have considered problems with

- integral constraints (Dido's problem)
- holonomic constraints (geodesics formulation)
- non-holonomic constraints (problems with higher derivatives)

But we have not considered inequality constraints

A problem

What is the shortest path, between A and B, avoiding an obstacle



E.G. what is the shortest path around a lake?

Formulation

Find extremals of
$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y')\, dx$$

subject to $y(0) = y_0$ and $y(1) = y_1$ and
$$y(x) \geq g(x)$$

Enforce the constraint by taking

$$y(x) = g(x) + z(x)^2$$

In other words introduce a “slack function” $z(x)$, and note that

$$y(x) - g(x) = z(x)^2 \geq 0$$

Formulation

We have slack function $z(x)$, and constraint $y(x) \geq g(x)$ and

$$\begin{aligned} y &= z^2 + g \\ y' &= 2zz' + g' \end{aligned}$$

Substitute these into the functional and we can change the original functional $F\{y\}$ for a new one in terms of $F\{z\}$

$$\begin{aligned} F\{y\} &= \int_{x_0}^{x_1} f(x, y, y')\, dx \\ F\{z\} &= \int_{x_0}^{x_1} f(x, z^2 + g, 2zz' + g')\, dx \end{aligned}$$

Euler-Lagrange equations

Given we look for the extremals of
$$F\{z\} = \int_{x_0}^{x_1} f(x, z^2 + g, 2zz' + g')\, dx$$

the Euler-Lagrange equations are
$$\begin{aligned} \frac{d}{dx}\frac{\partial f}{\partial z'} - \frac{\partial f}{\partial z} &= 0 \\ \frac{d}{dx}\left[2z\frac{\partial f}{\partial y'}\right] - 2z\frac{\partial f}{\partial y} - 2z'\frac{\partial f}{\partial y'} &= 0 \\ 2z\frac{d}{dx}\frac{\partial f}{\partial y'} + 2z'\frac{\partial f}{\partial y'} - 2z\frac{\partial f}{\partial y} - 2z'\frac{\partial f}{\partial y'} &= 0 \\ z\left[\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y}\right] &= 0 \end{aligned}$$

Euler-Lagrange equations

The Euler-Lagrange equations give

$$z\left[\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y}\right] = 0$$

for which there are two solutions

- **Euler areas:** The E-L equations are satisfied
- **Boundary areas:** $z(x) = 0$, so $y(x) = g(x)$ and the curve lies on the boundary.

Analogy: a global minima of function on an interval can happen at stationary point, or at the edges.
But we can mix the two along the curve y.

Example

Find the shortest path around a circular lake (radius a, centered at the origin), between the points $(b,0)$ and $(-b,0)$ (for $b > a$).

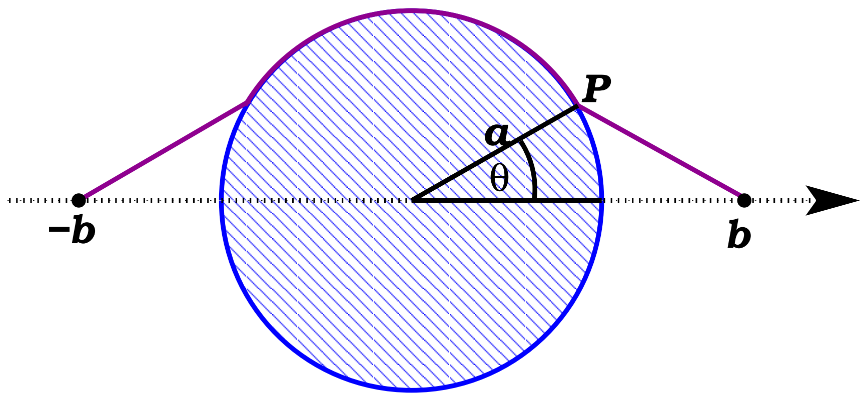
The conditions are

- **Euler areas:** The E-L equations are satisfied, so the curve is a straight line.
- **Boundary areas:** $z(x) = 0$, so $y(x) = g(x)$ and the curve lies on the boundary of the circle.

We can mix the two along the curve y.

Example

Given the conditions, the solution must look like



i.e. straight lines joining the end-points to a circular arc, where P, the point of intersection of the right-hand straight line, and the circle is at $(a\cos\theta, a\sin\theta)$.

Example

The total distance of such a line is

$$\begin{aligned} d(\theta) &= 2\sqrt{(b - a\cos\theta)^2 + a\sin^2\theta} + a(\pi - 2\theta) \\ &= 2\sqrt{b^2 - 2ab\cos\theta + a^2} + a(\pi - 2\theta) \end{aligned}$$

We find the minimum of $d(\theta)$, by differentiating WRT θ, to get

$$\begin{aligned} d' &= \frac{2ab\sin\theta}{\sqrt{b^2 - 2ab\cos\theta + a^2}} - 2a \\ &= 0 \end{aligned}$$

So

$$2ab\sin\theta = 2a\sqrt{b^2 - 2ab\cos\theta + a^2}$$

Example

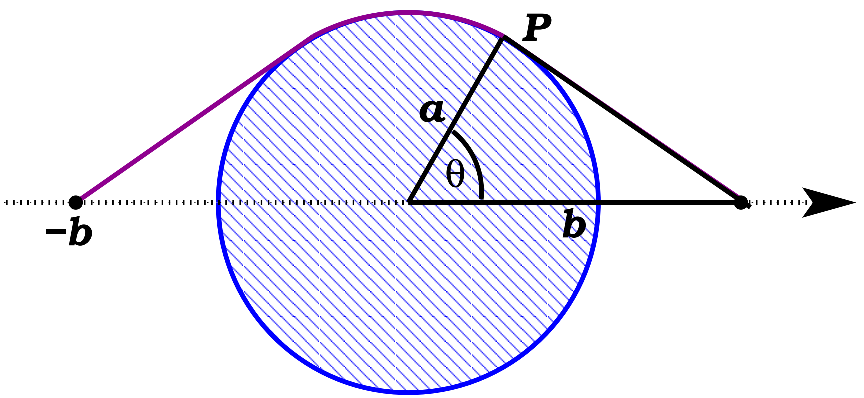
Dividing both sides by $2a$ we get the condition

$$\begin{aligned} b\sin\theta &= \sqrt{b^2 - 2ab\cos\theta + a^2} \\ b^2\sin^2\theta &= b^2 - 2ab\cos\theta + a^2 \\ b^2 - b^2\cos^2\theta &= b^2 - 2ab\cos\theta + a^2 \\ 0 &= b^2\cos^2\theta - 2ab\cos\theta + a^2 \\ 0 &= (b\cos\theta - a)^2 \end{aligned}$$

So the result is

$$\cos\theta = a/b$$

Example: solution



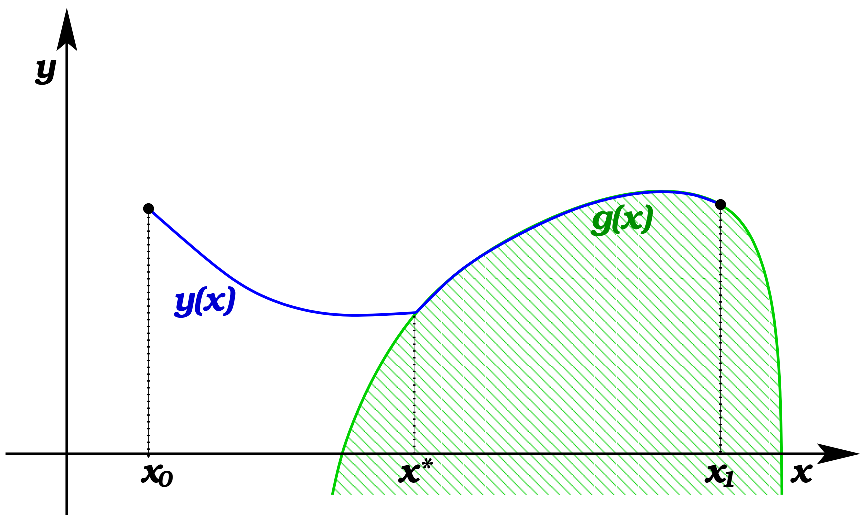
Think of what we would get if we stretch an elastic band between the two points.

General result

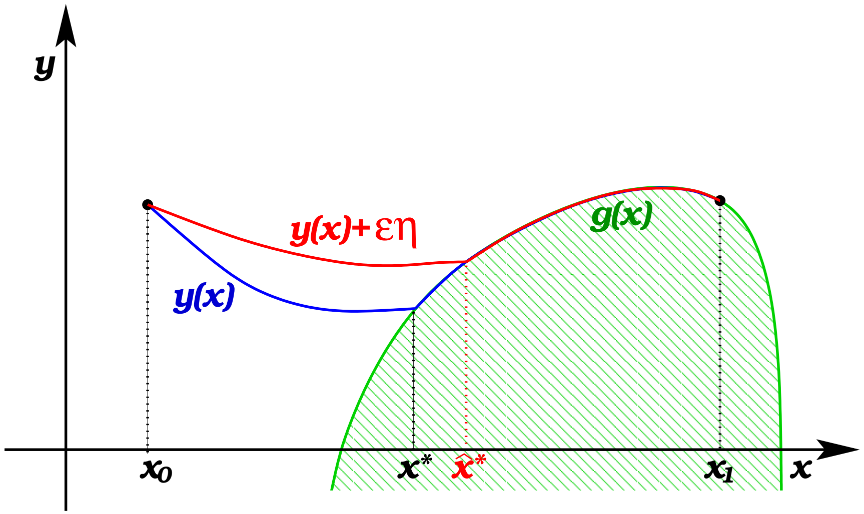
If $f_{y'}$ depends on y', then at the point where the extremal transfers from the Euler-Lagrange curve to the domain boundary the tangent varies continuously.

The problem is similar to that of the broken extremal. Here, the break is imposed by the change from one solution to the other (Euler-Lagrange to domain boundary). However, the condition can be seen in the same way, e.g. by perturbing the possible corner, along the boundary.

General result: proof



General result: proof



General result: proof

Similarly to the Weierstrass-Erdman Corner Conditions proof, we break the integral into two parts:

$$F\{y\} = F_1\{y\} + F_2\{y\} = \int_{x_0}^{x^*} f(x,y,y')\,dx + \int_{x^*}^{x_1} f(x,y,y')\,dx$$

but we will assume the shape of the curve on the RHS of x^* fits the boundary, e.g. $y(x) = g(x)$, and the LHS follows the E-L equations

$$F\{y\} = F_1\{y\} + F_2\{y\} = \int_{x_0}^{x^*} f(x,y,y')\,dx + \int_{x^*}^{x_1} f(x,g,g')\,dx$$

General result: proof

The first component of the first variation is, as with transversals, and corners conditions,

$$\delta F_1(\eta, y) \quad = \quad \lim_{\varepsilon\to\infty} \frac{1}{\varepsilon}\left[\int_{x_0}^{\hat{x}^*} f(x,\hat{y}_1,\hat{y}_1')\,dx - \int_{x_0}^{x^*} f(x,y_1,y_1')\,dx\right]$$

And as with corners, we get an integral term which results in the E-L equation, plus the additional constraint

$$\Big[p\delta y - H\delta x\Big]_{x^{*-}} - \Big[p\delta y - H\delta x\Big]_{x^{*+}} = 0$$

where

$$H \quad = \quad y'\frac{\partial f}{\partial y'} - f \quad \text{and} \quad p \quad = \quad \frac{\partial f}{\partial y'}$$

General result: proof

As with other potential corners, we get a corner condition

$$\Big[p\delta y - H\delta x\Big]_{x^{*-}} - \Big[p\delta y - H\delta x\Big]_{x^{*+}} = 0$$

but note that curve $y(x^*) = g(x^*)$, which constrains the end-point, so we cannot consider arbitrary variations $(\delta x, \delta y)$. In fact, we can only consider variations where

$$\delta y = g'\delta x$$

Assuming that dg/dx is in fact defined the above is

$$\Big[pg'\delta x - H\delta x\Big]_{x^{*-}} - \Big[pg'\delta x - H\delta x\Big]_{x^{*+}} = 0$$

General result: proof

The condition

$$\left[pg'\delta x - H\delta x\right]_{x^*-} - \left[pg'\delta x - H\delta x\right]_{x^*+} = 0$$

which can be simplified to

$$\left[pg' - H\right]_{x^*-} - \left[pg' - H\right]_{x^*+} = 0$$

Substituting H and p, and $y' = g'$ on the RHS of x^* we get

$$\left[g'\frac{\partial f}{\partial y'} - y'\frac{\partial f}{\partial y'} + f\right]_{x^*-} - \left[g'\frac{\partial f}{\partial y'} - g'\frac{\partial f}{\partial y'} + f\right]_{x^*+} = 0$$

General result: proof

Simplifying we get

$$\left[(g' - y')\frac{\partial f}{\partial y'} - f\right]_{x^*-} + [f]_{x^*+} = 0$$

or

$$\left[(g' - y')\frac{\partial f}{\partial y'}\right]_{x^*-} - [f]_{x^*-} + [f]_{x^*+} = 0$$

- Consider the term $-\{[f]_{x^*-} - [f]_{x^*+}\}$
- Note that at the "join" $y(x^*) = g(x^*)$, so if the two limits of f differ it is because of a difference in y' on either side of the join
- Treat f as a function of just y', i.e, $f(x,y,y') = q_{x,y}(y')$

General result: proof

Taking $q_{x,y}(y') = f(x,y,y')$ where

- on the left side of x^*, we have y' determined by E-L equations
- on the right side of x^* we have $y' = g'$

So

$$\begin{aligned} [f]_{x^*-} - [f]_{x^*+} &= \lim_{x \to x^*-} f(x,y,y') - \lim_{x \to x^*+} f(x,g,g') \\ &= q_{x^*,y^*}[y'(x^*)] - q_{x^*,y^*}[g'(x^*)] \end{aligned}$$

Given its all the same, I won't keep writing the subscripts of q, and will just use

$$q(z) = q_{x^*,y^*}(z)$$

General result: proof

The Mean Value Theorem states: *if a function $q(z)$ is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then there exists a point c in (a,b) such that*

$$q(b) - q(a) = (b-a)q'(c)$$

So we get

$$\begin{aligned} [f]_{x^*-} - [f]_{x^*+} &= q(y'(x^*)) - q(g'(x^*)) \\ &= [y'(x^*) - g'(x^*)]\,q'(c) \end{aligned}$$

for some c between $g'(x^*)$ and $y'(x^*)$

General result: proof

Taking $q(y') = f(x,y,y')$ we get

$$\frac{d}{dz} q(z) = \frac{\partial f}{\partial y'}(x,y,y') \bigg|_{y'=z}$$

So

$$q'(c) = \frac{\partial f}{\partial y'}(x^*,y^*,c)$$

and hence

$$\begin{aligned} [f]_{x^{*-}} - [f]_{x^{*+}} &= q(y'(x^*)) - q(g'(x^*)) \\ &= [y'(x^*) - g'(x^*)]\, q'(c) \\ &= [y'(x^*) - g'(x^*)] \frac{\partial f}{\partial y'}(x^*,y^*,c) \end{aligned}$$

General result: proof

So the condition from before can be rewritten as follows:

$$\begin{aligned} \left[(g' - y') \frac{\partial f}{\partial y'} - f \right]_{x^{*-}} + [f]_{x^{*+}} &= 0 \\ \left[(g' - y') \left(\frac{\partial f}{\partial y'}(x,y,y') - \frac{\partial f}{\partial y'}(x,y,c) \right) \right]_{x^*} &= 0 \end{aligned}$$

for some c between $g'(x^*)$ and $y'(x^*)$

General result: proof

$$(g'(x^*) - y'(x^*)) \left(\frac{\partial f}{\partial y'}(x^*, y(x^*), y'(x^*)) - \frac{\partial f}{\partial y'}(x^*, y(x^*), c) \right) = 0$$

So there are two possibilities:

- $g'(x^*) = y'(x^*)$, which means that y meets the boundary at a tangent to the boundary.
- $\frac{\partial f}{\partial y'}(x,y,y') - \frac{\partial f}{\partial y'}(x,y,c) = 0$. This latter condition holds when $\frac{\partial f}{\partial y'}$ is constant with respect to y', i.e.,

$$\frac{\partial^2 f}{\partial y'^2} = 0$$

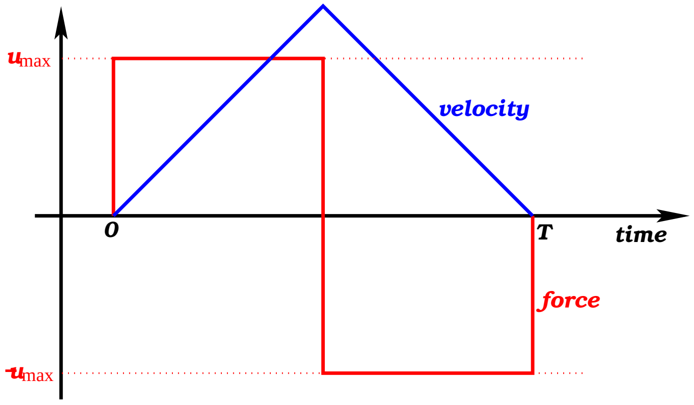
In the lake example, $\frac{\partial^2 f}{\partial y'^2} \neq 0$

Example: parking a car

- Revisit the problem of parking a car.
- If we think about the problem, it makes no sense unless there is maximum force $u_{\max}$.
    - otherwise we move from A to B arbitrarily fast.
- There are no valid E-L equation solutions.
- We must end-up in the boundary domain, e.g. $u = \pm u_{\max}$
    - obvious solution is to accelerate as fast as possible until we get half-way, and then to decelerate as fast as possible.
    - $\frac{\partial f}{\partial \dot{u}} = 0$, so we don't have to stress about continuity (u is not continuous either)

Example: parking a car

- Our solution is in the boundary domain, e.g. $u = \pm u_{\max}$



- called a **bang-bang controller**

Bang-bang controllers

Bang-bang controllers appear in a number of other contexts, and we will consider them in more generality later.