Variational Methods & Optimal Control

lecture 23

Matthew Roughan
<matthew.roughan@adelaide.edu.au>

Discipline of Applied Mathematics
School of Mathematical Sciences
University of Adelaide

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More Optimal Control Examples

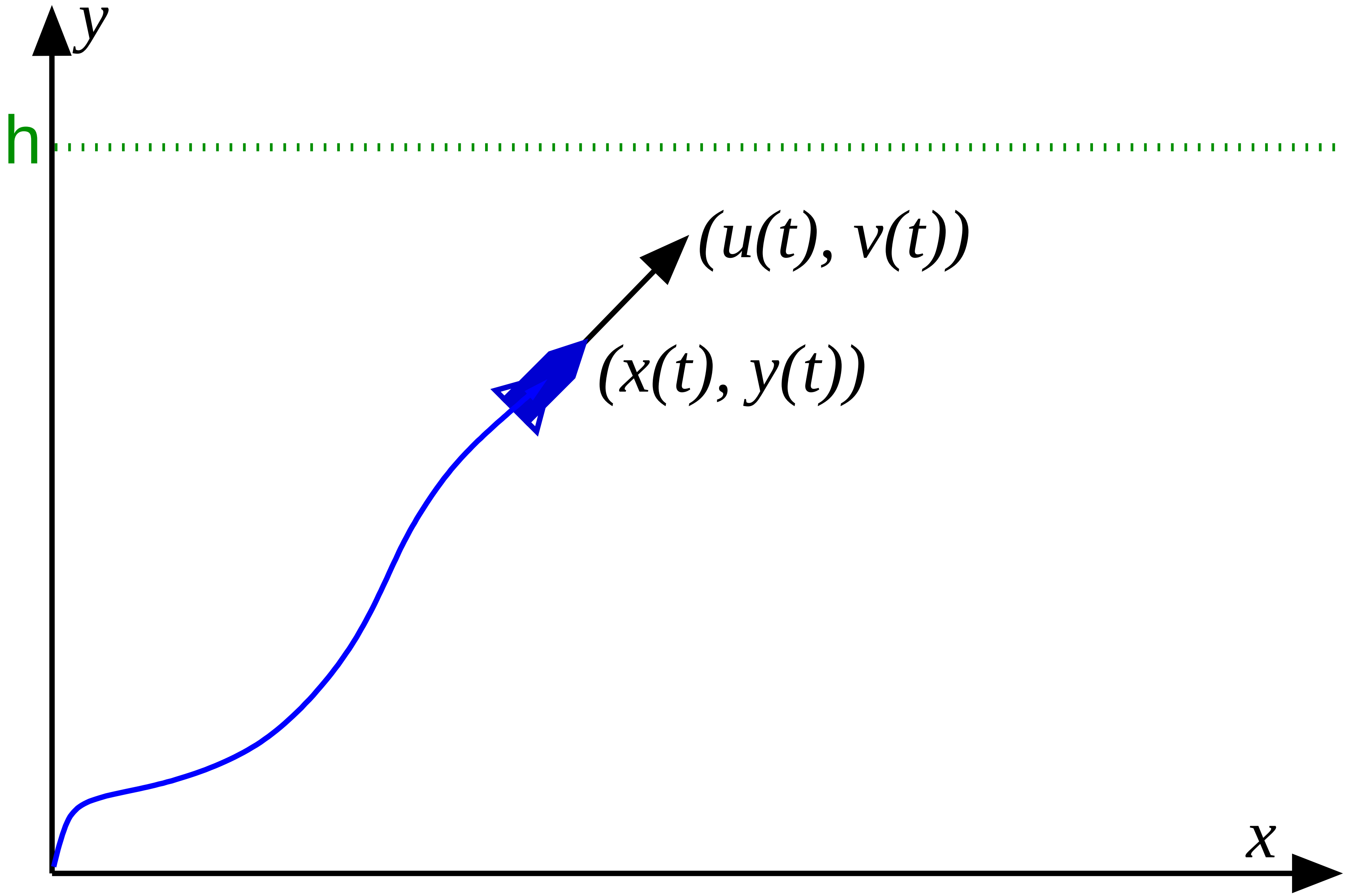
An aerospace example: a rocket launch profile.

Example: launching a rocket

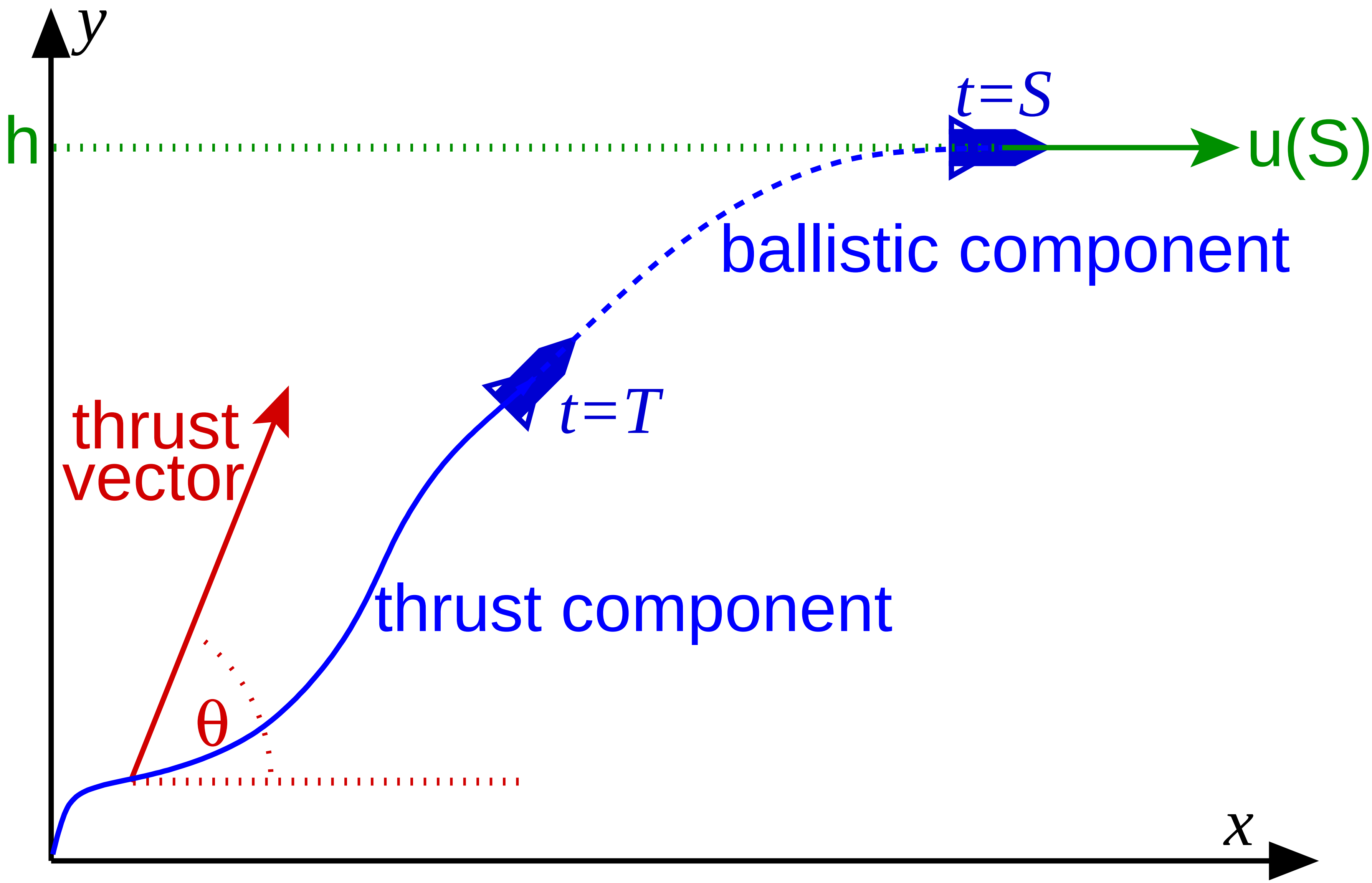
Launch a rocket (with one stage) to deliver its payload into Low-Earth Orbit (LEO) at some height h above the Earth's surface.
Assumptions:

- ignore drag, and curvature and rotation of Earth
- LEO so assume gravitational force at ground and orbit are approximately the same
- thrust will generate acceleration a, which is predefined by rocket parameters
- we thrust for some time T, then follow a ballistic trajectory until (hopefully) we reach height h, at zero vertical velocity, and with horizontal velocity matching the required orbital injection speed.

Example: launching a rocket



Example: launching a rocket



Example: launching a rocket

Notation:

$$\begin{aligned} x &= \text{horizontal position} \\ y &= \text{vertical position} \\ u &= \text{horizontal velocity} \\ v &= \text{vertical velocity} \end{aligned}$$

Initial conditions $x(0) = y(0) = u(0) = v(0) = 0$. Thrust stops at time T, and then at some later time S, we reach the peak of the trajectory where

$$\begin{aligned} y(S) &= h \\ u(S) &= u_o, \text{ orbital velocity} \\ v(S) &= 0 \end{aligned}$$

We don't actually care about the final position $x(S)$

Example: launching a rocket

- Control: thrust profile is pre-determined. The only thing we can control (in this problem) is the **angle** of thrust.
  - Thrust $a(t)$ is constant for our example.
  - Measure the angle of thrust $\theta(t)$ relative to horizontal.
- want to minimize fuel
  - but this is equivalent to minimizing time, e.g.,

$$F = \int_0^t a\, dt = a \int_0^T 1\, dt$$

- need to get to height h
- need to get to horizontal velocity u_o to enter orbit

Constraint equations

ust component: $t \leq T$

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{u} &= a\cos\theta \\ \dot{v} &= a\sin\theta - g \end{aligned}$$

ial point:
$) = y(0) = u(0) = v(0) = 0.$

al point: $free$

Ballistic component: $T < t \leq S$

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{u} &= 0 \\ \dot{v} &= -g \end{aligned}$$

Initial point: fixed
$x(T), y(T), u(T), v(T)$

Final point:
$x(S)$ $free$,
$y(S) = h$, $v(S) = 0$, $u(S) = u_o$

1st consider ballistic component

For $t \in [T,S]$ we have no control, and

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{u} &= 0 \\ \dot{v} &= -g \end{aligned}$$

we can calculate the top of the resulting parabola as

$$\begin{aligned} u(S) &= u(T) \\ v(S) &= 0 \\ y(S) &= y(T) + v(T)^2/2g \end{aligned}$$

and $x(T)$ and $x(S)$ are free.

Example: co-ordinate transform

So we can change variables: make the final point $t = T$, and take variables u, v as before, and

$$z = y + v^2/2g.$$

We can differentiate this and combine with previous results to get the new **system DEs**

$$\begin{aligned}
\dot{u} &= a\cos\theta \\
\dot{v} &= a\sin\theta - g \\
\dot{z} &= \dot{y} + v\dot{v}/g \\
&= v(1 + \dot{v}/g) \\
&= \frac{av}{g}\sin\theta
\end{aligned}$$

Example: optimization functional

Time minimization problem

$$T = \int_0^T 1 \, dt$$

Including Lagrange multipliers for the 3 system constraints we aim to minimize

$$J\{\theta\} = \int_0^T 1 + \lambda_u \left(\dot{u} - a\cos\theta\right) + \lambda_v \left(\dot{v} - a\sin\theta + g\right) + \lambda_z \left(\dot{z} - \frac{av}{g}\sin\theta\right) dt$$

subject to

$$\begin{aligned} u(0) &= 0, & u(T) &= u_o \\ v(0) &= 0, & v(T) &= free \\ z(0) &= 0, & z(T) &= h \\ \theta(0) &= free, & \theta(T) &= free \end{aligned}$$

Example: Euler-Lagrange equations

E-L equations

$$
\begin{array}{rlcccl}
u: & \dfrac{\partial h}{\partial u} - \dfrac{d}{dt}\dfrac{\partial h}{\partial \dot{u}} & = & 0 & \Rightarrow & \dot{\lambda}_u = 0 \\
v: & \dfrac{\partial h}{\partial v} - \dfrac{d}{dt}\dfrac{\partial h}{\partial \dot{v}} & = & 0 & \Rightarrow & \dot{\lambda}_v = -\lambda_z \frac{a}{g} \sin\theta \\
z: & \dfrac{\partial h}{\partial z} - \dfrac{d}{dt}\dfrac{\partial h}{\partial \dot{z}} & = & 0 & \Rightarrow & \dot{\lambda}_z = 0 \\
\theta: & \dfrac{\partial h}{\partial \theta} - \dfrac{d}{dt}\dfrac{\partial h}{\partial \dot{\theta}} & = & 0 & \Rightarrow &
\end{array}
$$

$$
a\lambda_u \sin\theta - \lambda_v a \cos\theta - \lambda_z \frac{av}{g} \cos\theta = 0
$$

(λ equations give back systems DEs)

Example: solving the E-L equations

Take the v equation, and noting that $\dot{v} = a \sin\theta - g$

$$\begin{aligned}
\dot{\lambda}_v &= -\lambda_z \frac{a}{g} \sin\theta \\
&= -\frac{\lambda_z}{g}(\dot{v} + g) \\
\lambda_v &= -\frac{\lambda_z}{g}(v + gt + c) \\
&= -\frac{\lambda_z v}{g} - \lambda_z t + b
\end{aligned}$$

Example: solving the E-L equations

Substitute

$$\lambda_v = -\frac{\lambda_z v}{g} - \lambda_z t + b$$

into the θ E-L equation (dropping the common factor a)

$$\lambda_u \sin\theta - \lambda_v \cos\theta - \lambda_z \frac{v}{g} \cos\theta = 0$$

and we get

$$\begin{aligned}
\lambda_u \sin\theta + \left(\frac{\lambda_z v}{g} + \lambda_z t - b\right)\cos\theta - \lambda_z \frac{v}{g}\cos\theta &= 0 \\
\lambda_u \sin\theta + (\lambda_z t - b)\cos\theta &= 0 \\
\tan\theta &= -(\lambda_z t - b)/\lambda_u
\end{aligned}$$

Example: solution

Remember that λ_u and λ_v and b are all constants, so the equation

$$\boxed{\tan\theta = -\left(\lambda_z t - b\right)/\lambda_u}$$

- angle of thrust now specified

$$\theta = \tan^{-1}\left(-\left(\lambda_z t - b\right)/\lambda_u\right)$$

- but we need to determine constants

Example: end-point conditions

Final end-points conditions

$$
\begin{aligned}
T &= free \\
z(T) &= h \\
u(T) &= u_o, \text{ orbital velocity} \\
v(T) &= free \\
\theta(T) &= free \\
\lambda_u &= free \\
\lambda_v &= free \\
\lambda_z &= free
\end{aligned}
$$

Example: natural boundary conditions

The free-end point boundary condition for

$$F\{t,\mathbf{q},\dot{\mathbf{q}}\} = \int L(t,\mathbf{q},\dot{\mathbf{q}})\,dt$$

is

$$\sum_{k=1}^{n} p_k \delta q_k - H\delta t = 0 \text{ where } p_k = \frac{\partial L}{\partial \dot{q}_k} \text{ and } H = \sum_{k=1}^{n} \dot{q}_k p_k - L$$

In this problem

$$\frac{\partial L}{\partial \dot{\lambda}_k} = 0, \quad \frac{\partial L}{\partial \dot{\theta}} = 0, \quad \frac{\partial L}{\partial \dot{u}} = \lambda_u, \quad \frac{\partial L}{\partial \dot{v}} = \lambda_v, \quad \frac{\partial L}{\partial \dot{z}} = \lambda_z$$

Example: natural boundary conditions

Consider δq_k for each co-ordinate:

- for fixed co-ordinates u and z, we have $\delta q_k = 0$
- its free for θ, λ_u, λ_v, λ_z, but in each case the corresponding $p_k = 0$, so we can ignore these.
- only case where it matters is δv, which we can vary, and for which $p_v = \lambda_v$.

Also δt is free, so we get two end-point conditions at $t = T$.

$$\begin{aligned} H(T) &= 0 \\ p_v = \lambda_v(T) &= 0 \end{aligned}$$

Example: natural boundary conditions

Given $\lambda_v(T) = 0$, and from previous work

$$\lambda_v = -\frac{\lambda_z v}{g} - \lambda_z t + b$$

we get

$$\begin{aligned}
\lambda_z v(T)/g &= -\lambda_z T + b \\
&= \lambda_u \tan\theta(T) \\
v(T) &= \frac{\lambda_u g}{\lambda_z} \tan\theta(T)
\end{aligned}$$

Example: natural boundary conditions

$$\frac{\partial L}{\partial \dot{\lambda}_k} = 0, \quad \frac{\partial L}{\partial \dot{\theta}} = 0, \quad \frac{\partial L}{\partial \dot{u}} = \lambda_u, \quad \frac{\partial L}{\partial \dot{v}} = \lambda_v, \quad \frac{\partial L}{\partial \dot{z}} = \lambda_z$$

So H is given by

$$H = \lambda_u \dot{u} + \lambda_v \dot{v} + \lambda_z \dot{z} - L$$

Substitute L, and the system DEs, and we get

$$H = \lambda_u \dot{u} + \lambda_v \dot{v} + \lambda_z \dot{z} - 1$$

The end-point condition at $t = T$ is therefore

$$\lambda_u \dot{u} + \lambda_v \dot{v} + \lambda_z \dot{z} = 1$$

Example: natural boundary conditions

Substitute

$$\begin{aligned}
\lambda_v &= -\lambda_z v/g - \lambda_z t + b \\
&= -\lambda_z v/g + \lambda_u \tan\theta \\
\dot{u} &= a\cos\theta \\
\dot{v} &= a\sin\theta - g \\
\dot{z} &= \frac{av}{g}\sin\theta
\end{aligned}$$

Into

$$\lambda_u \dot{u} + \lambda_v \dot{v} + \lambda_z \dot{z} = 1$$

and we get

Example: natural boundary conditions

We get

$$
\begin{aligned}
\lambda_u \dot{u} + \lambda_v \dot{v} + \lambda_z \dot{z} &= 1 \\
\lambda_u a \cos\theta + (-\lambda_z v/g + \lambda_u \tan\theta)(a\sin\theta - g) + \lambda_z \frac{av}{g}\sin\theta &= 1 \\
\lambda_u a \cos\theta + \lambda_z v + \lambda_u a \tan\theta \sin\theta - g\lambda_u \tan\theta &= 1 \\
\lambda_u a\,(\cos\theta + \tan\theta\sin\theta) + \lambda_z v - g\lambda_u \tan\theta &= 1 \\
\lambda_u a \left(\frac{\cos^2\theta + \sin^2\theta}{\cos\theta} \right) + \lambda_z v - g\lambda_u \tan\theta &= 1 \\
\lambda_u a \sec\theta + \lambda_z v - g\lambda_u \tan\theta &= 1
\end{aligned}
$$

all evaluated at $t = T$. Combine with $g\lambda_u \tan\theta = \lambda_z v$ and

$$\lambda_z = \cos(\theta(T))/a$$

Example: natural boundary conditions

Another way to get the same result is to note

$$H = \lambda_u \dot{u} + \lambda_v \dot{v} + \lambda_z \dot{z} - L$$

and

$$L = 1 + \lambda_u \left(\dot{u} - a\cos\theta\right) + \lambda_v \left(\dot{v} - a\sin\theta + g\right) + \lambda_z \left(\dot{z} - \frac{av}{g}\sin\theta\right)$$

so

$$H = \lambda_u a\cos\theta + \lambda_v[a\sin\theta - g] + \frac{av\lambda_z}{g}\sin\theta - 1$$

which is what we got near the start of the previous slide before substituting $\lambda_v = -\lambda_z v/g + \lambda_u \tan\theta$.

Example: natural boundary conditions

At the starting point, all of the co-ordinates are fixed (except for θ, and the Lagrange multipliers), so the only free-end points condition at this point is

$$H = 0$$

as before. In fact, if $a = const$ the problem is not time-dependent, so H is conserved, i.e.

$$H(t) = 0$$

for the entire rocket flight. Note though, that for this system, H is not "energy" as this is not conserved (unless you include the chemical energy stored in the rocket).

Example: acceleration profile

The next steps depend on the acceleration profile $a(t)$, but lets take a simple case $a = const$.

First we can solve the DEs, with respect to θ using the chain rule

$$\frac{dX}{dt} = \frac{dX}{d\theta}\frac{d\theta}{dt} = -\cos^2\theta \frac{\lambda_z}{\lambda_u}\frac{dX}{d\theta}$$

e.g. from the system DE $\dot{u} = a\cos\theta$

$$\begin{aligned}
\dot{u} &= -\cos^2\theta \frac{\lambda_z}{\lambda_u}\frac{du}{d\theta} \\
\frac{du}{d\theta} &= -\frac{\lambda_u}{\lambda_z\cos^2\theta}\dot{u} \\
&= -\frac{a\lambda_u}{\lambda_z\cos\theta}
\end{aligned}$$

Example: acceleration profile

$$\frac{dX}{d\theta} = \frac{dX}{dt} \Big/ \frac{d\theta}{dt} = \frac{dX}{dt} \Big/ \left(-\cos^2\theta \frac{\lambda_z}{\lambda_u} \right)$$

The complete set of system DEs becomes

$$\begin{aligned}
\frac{du}{d\theta} &= -\frac{a\lambda_u}{\lambda_z \cos\theta} \\
\frac{dv}{d\theta} &= -\frac{a\lambda_u}{\lambda_z} \frac{\sin\theta}{\cos^2\theta} + \frac{g\lambda_u}{\lambda_z \cos^2\theta} \\
\frac{dz}{d\theta} &= -\frac{a\lambda_u}{g\lambda_z} \frac{\sin\theta}{\cos^2\theta} v(\theta)
\end{aligned}$$

These can just be integrated with respect to θ

Example: acceleration profile

The system DEs can be directly integrated (with respect to θ) including initial conditions $u(0) = v(0) = z(0) = 0$ to get

$$
\begin{aligned}
u(\theta) &= \frac{a\lambda_u}{\lambda_z} \log\left(\frac{\sec\theta_0 + \tan\theta_0}{\sec\theta + \tan\theta}\right) \\
v(\theta) &= \frac{a\lambda_u}{\lambda_z} (\sec\theta_0 - \sec\theta) - \frac{g\lambda_u}{\lambda_z} (\tan\theta_0 - \tan\theta) \\
z(\theta) &= \frac{a^2\lambda_u^2}{g\lambda_z^2} \sec\theta_1 (\sec\theta_0 - \sec\theta) - \frac{a^2\lambda_u^2}{2g\lambda_z^2} (\tan^2\theta_0 - \tan^2\theta) \\
&\quad + \frac{a\lambda_u^2}{2\lambda_z^2} \left[\tan\theta_0 \sec\theta_0 - \tan\theta\sec\theta + \log\left(\frac{\sec\theta_0 + \tan\theta_0}{\sec\theta + \tan\theta}\right)\right] \\
\theta &= \tan^{-1}\left(-\left(\lambda_z t - b\right)/\lambda_u\right)
\end{aligned}
$$

Example: calculating the constants

There are five constants to calculate:

- θ_0 the initial angle of thrust
- θ_1 the final angle of thrust
- λ_u
- λ_z
- b

and we also need to calculate T.

Solving for end-point conditions is non-trivial, but a method that works well (from Lawden) follows.

Example: calculating the constants

Take the equation for v at time T, and substitute $\lambda_z v(T) = g\lambda_u \tan\theta_1$ to get

$$
\begin{aligned}
v(\theta_1) &= \frac{a\lambda_u}{\lambda_z}(\sec\theta_0 - \sec\theta_1) - \frac{g\lambda_u}{\lambda_z}(\tan\theta_0 - \tan\theta_1) \\
\frac{g\lambda_u}{\lambda_z}\tan\theta_1 &= \frac{a\lambda_u}{\lambda_z}(\sec\theta_0 - \sec\theta_1) - \frac{g\lambda_u}{\lambda_z}(\tan\theta_0 - \tan\theta_1) \\
\sec\theta_1 &= \sec\theta_0 - \frac{g}{a}\tan\theta_0
\end{aligned}
$$

which gives us a way to calculate θ_1 from θ_0. Once we know θ_1 we can calculate λ_u using $\lambda_u a = \cos\theta_1$, and b from $\tan\theta = (-(\lambda_z t - b)/\lambda_u)$ at $t = 0$. Then we can calculate λ_z from $u(\theta_1) = u_o$, the orbital injection velocity

Example: calculating the constants

So the only remaining question is how to calculate θ_0. We do so numerically, by

- take a range of θ_0
- calculate all of the above
- use this to calculate $z(T) = z_1$ as a function of θ_0
- look for the point where $z_1(\theta_0) = h$ the orbit height.

That gives us the θ_0, from which we can derive everything else. There are good numerical methods to search for such a solution, particularly if we start with a clear range over which to look.

Example: restricting choice of θ_0

Calculating the range of θ_0 to search

- The maximum (reasonable) value for θ_0 is $\pi/2$.
- The minimum value of θ_0 will be determined by the minimum possible value of θ_1, i.e., $\theta_1 = 0$

$$
\begin{aligned}
\sec\theta_1 &= \sec\theta_0 - \frac{g}{a}\tan\theta_0 \\
\sec 0 &= \sec\theta_0 - \frac{g}{a}\tan\theta_0 \\
1 &= \sec\theta_0 - \frac{g}{a}\tan\theta_0 \\
1 &= \frac{1+\tan^2\theta_0/2}{1-\tan^2\theta_0/2} - \frac{g}{a}\frac{2\tan\theta_0/2}{1-\tan^2\theta_0/2} \\
1-\tan^2\theta_0/2 &= 1+\tan^2\theta_0/2 - \frac{2g}{a}\tan\theta_0/2
\end{aligned}
$$

Example: restricting choice of θ_0

$$\begin{aligned}
1 - \tan^2\theta_0/2 &= 1 + \tan^2\theta_0/2 - \frac{2g}{a}\tan\theta_0/2 \\
2\tan^2\theta_0/2 - \frac{2g}{a}\tan\theta_0/2 &= 0 \\
\tan\theta_0/2\left(\tan\theta_0/2 - \frac{g}{a}\right) &= 0
\end{aligned}$$

Now θ_0 can't be zero, so the last step implies that the minimum value of θ_0 is

$$\theta_0 = 2\tan^{-1}(g/a)$$

Note the existence of a minimum critical h below which we can't find a trajectory of this type.

Example: parameters

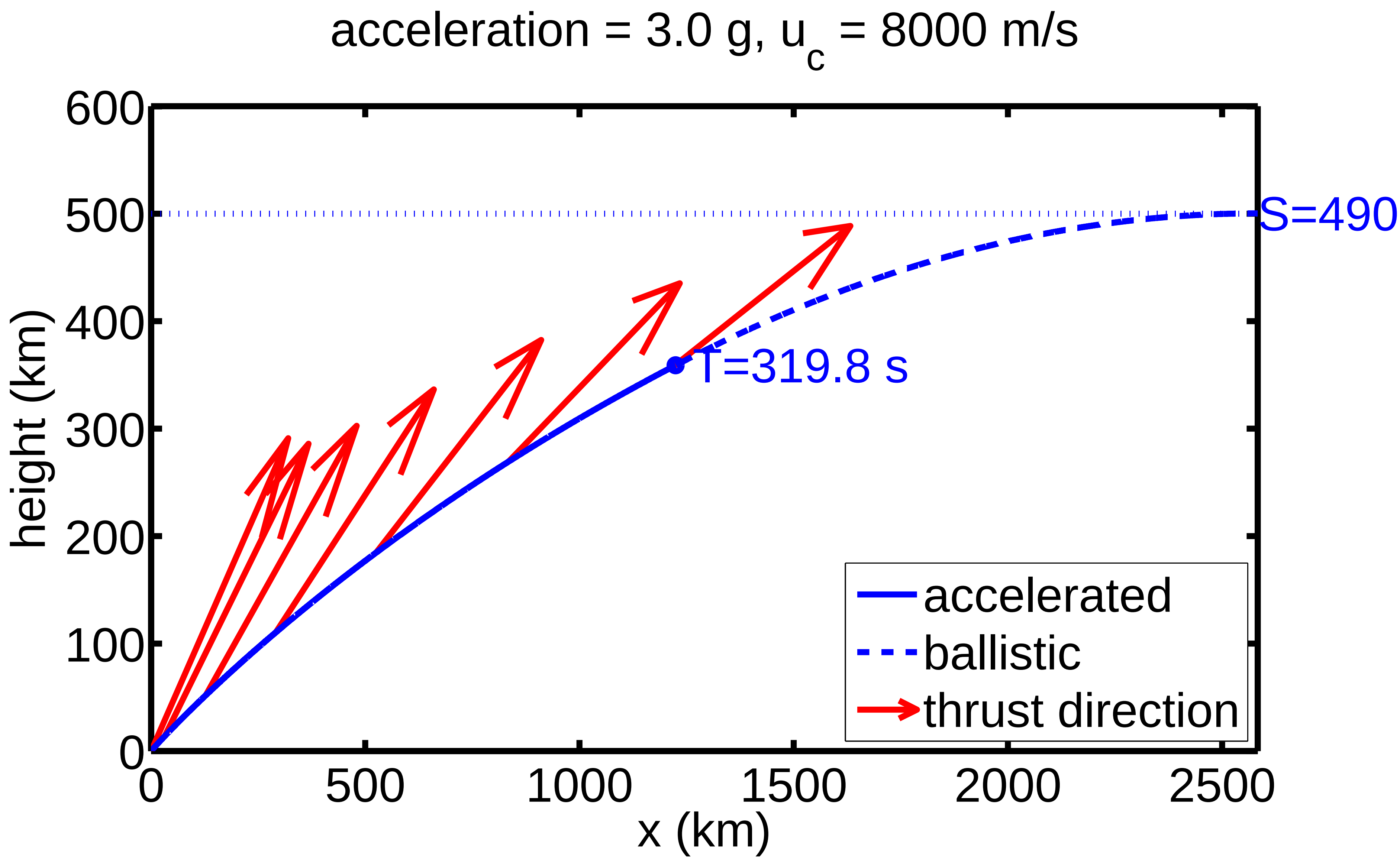
Parameters of previous example consistent with a LEO.

$$\begin{aligned} h &= 500 \text{ km} \\ u_o &= 8000 \text{ m/s} \\ g &= 9.8 \text{ m/s}^2 \\ a &= 3g \end{aligned}$$

Derived constants

$$\begin{aligned} \theta_0 &= 0.2349\pi & \theta_1 &= 0.0973\pi \\ \lambda_u &= 0.0324 & \lambda_z &= 6.0257e-05 \\ b &= -0.0295 \\ T &= 319.8 \text{ seconds} \\ S &= 489.6 \text{ seconds} \end{aligned}$$

Example: trajectory



Example: generalizations

More realistic assumptions

- non-zero drag (depends on velocity and height)
- thrust is constant, but rocket mass changes, so that acceleration isn't constant
- multiple stages
- centripetal forces

For more examples, and discussion see Lawden, "Optimal Trajectories for Space Navigation", Butterworths, 1963 (which is incidentally where the above example comes from).