Tutorial 1 Solutions

1. See me if you have problems with any of these questions.

2. Use the multivariable chain rule to find dz/dt where

$$z = 2x^2 + 3xy - 4y^2$$

and

$$x = \cos t, \quad \text{and } y = \sin t.$$

Solution: The multivariable chain rule states that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

and we can calculate

$$\begin{aligned}
\frac{\partial z}{\partial x} &= 4x + 3y \\
\frac{\partial z}{\partial y} &= 3x - 8y \\
\frac{dx}{dt} &= -\sin t \\
\frac{dy}{dt} &= \cos t
\end{aligned}$$

Substituting these into the chain rule we get

$$\frac{dz}{dt} = -\sin t(4x + 3y) + \cos t(3x - 8y).$$

We can further simplify by substituting the parametric form of x and y into the above expression to get

$$\frac{dz}{dt} = 3\cos^2 t - 12 \sin t \cos t - 3\sin^2 t = 3x^2 - 12xy - 3y^2.$$

We could also have solved the problem by substituting the parametric forms of x and y into the formula for z, and then taking the derivative, but often this is more complicated.

3. Use Taylor's Theorem to derive a polynomial approximation for $f(x, y) = \sin(x+y^2)$.

Solution: The first and second partial derivatives of f are

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \cos(x + y^2) \\
\frac{\partial f}{\partial y} &= 2y\cos(x + y^2) \\
\frac{\partial^2 f}{\partial x^2} &= -\sin(x + y^2) \\
\frac{\partial^2 f}{\partial x \partial y} &= -2y\sin(x + y^2) \\
\frac{\partial^2 f}{\partial y^2} &= 2\cos(x + y^2) - 4y^2\sin(x + y^2)
\end{aligned}$$

At the point $(x, y) = (0, 0)$, these partial derivatives are

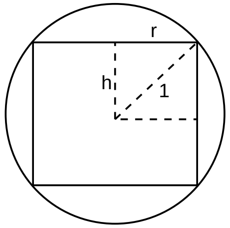
$$\begin{aligned}
\frac{\partial f}{\partial x} &= 1 \\
\frac{\partial f}{\partial y} &= 0 \\
\frac{\partial^2 f}{\partial x^2} &= 0 \\
\frac{\partial^2 f}{\partial x \partial y} &= 0 \\
\frac{\partial^2 f}{\partial y^2} &= 2
\end{aligned}$$

So the Taylor Polynomial about $(0, 0)$ is

$$\begin{aligned}
f(x, y) \quad &= \quad f(0, 0) + x \left.\frac{\partial f}{\partial x}\right|_{(0,0)} + y \left.\frac{\partial f}{\partial y}\right|_{(0,0)} \\
&\quad + \frac{1}{2}\left[x^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} + 2xy \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)} + y^2 \left.\frac{\partial^2 f}{\partial y^2}\right|_{(0,0)}\right] + \cdots \\
simeq &\quad x + y^2
\end{aligned}$$

4. Find the cylinder of largest volume that can be placed inside a sphere of radius 1.

Solution: The figure below shows a cross-section of the cylinder inscribed in the sphere. Note that there is no value in making the cylinder small than it can be, so the the top and bottom rings of the cylinder, being circles, will sit against the surface of the sphere. Viewed from the side, the radius r of the circles at the top and bottom if the cylinder is half the length of the horizontal edge of the rectangle shown. The height of the cylinder $H = 2h$ is also shown.



We can immediately see from the figure that there is a right-angled triangle formed by h, r and 1, and so it is easy to express the optimization problem as

$$\max_{r,h} V = 2h\pi r^2,$$

such that

$$r^2 + h^2 = 1.$$

We include the constraint in the optimization by simply taking a new objective including a Lagrange multiplier, i.e.,

$$\max_{r,h} F = 2h\pi r^2 + \lambda(r^2 + h^2 - 1).$$

We maximize by setting the partial derivatives to zero, being careful to include all three variables:

$$\begin{aligned} \frac{\partial F}{\partial r} &= 0, \\ \frac{\partial F}{\partial h} &= 0, \\ \frac{\partial F}{\partial \lambda} &= 0. \end{aligned}$$

These give

$$\begin{aligned} 2(2h\pi + \lambda)r &= 0, \\ 2(\pi r^2 + \lambda h) &= 0, \\ r^2 + h^2 - 1 &= 0. \end{aligned}$$

As $r \neq 0$ for $F > 0$, the first equation implies that

$$h = -\frac{\lambda}{2\pi}.$$

We can substitute this into the second equation to get

$$\begin{aligned} \pi r^2 &= -\lambda h \\ &= \frac{\lambda^2}{2\pi} \\ r^2 &= \frac{\lambda^2}{2\pi^2} \end{aligned}$$

We take the positive root here for convenience so $r = \lambda/\pi\sqrt{2}$. Substituting these into the third equation we get

$$\begin{aligned} r^2 + h^2 &= 1 \\ \frac{\lambda^2}{2\pi^2} + \frac{\lambda^2}{4\pi^2} &= 1 \\ \lambda^2\left[\frac{1}{2} + \frac{1}{4}\right] &= \pi^2 \\ \lambda^2 &= \frac{4}{3}\pi^2. \end{aligned}$$

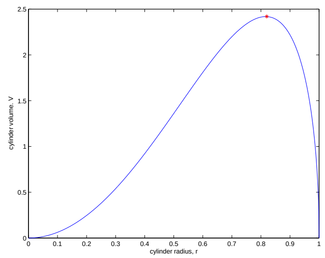
The final result is therefore

$$\begin{aligned} r &= \sqrt{\frac{2}{3}} \simeq 0.82 \\ h &= \sqrt{\frac{1}{3}} \simeq 0.58 \end{aligned}$$

so the cylinder is broader than it is tall, and has volume

$$V = 2h\pi r^2 \simeq 2.42.$$

We can argue that this must be a maximum from the fact that the minimum volume cylinder would have volume zero, which is not the case here, however, the following figure shows a simple numerical plot of the volume of the inscribed cylinder as a function of r.



Compare this to the volume of the sphere itself

$$V_S = \frac{4}{3}\pi R^2 = 4.19.$$