Tutorial 2: Wednesday 15th August

Basic Euler-Lagrange equation problems.

1. Find the extremals of the functionals below subject to the fixed end point conditions prescribed.

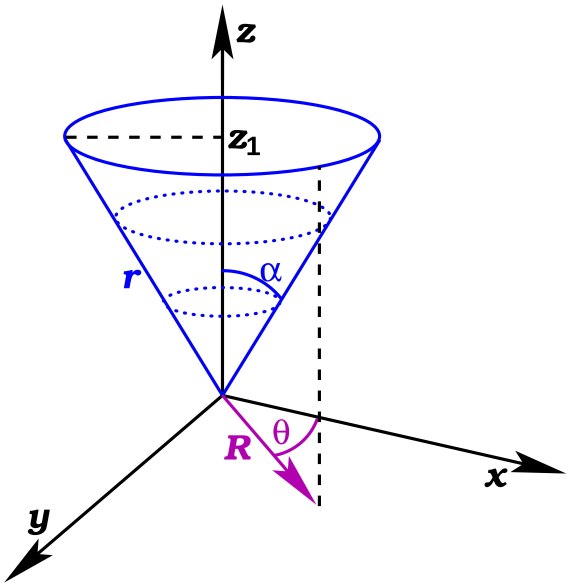
   (a). $\displaystyle\int_0^{\pi/2} \left(y^2 + y'^2 - 2y \sin x\right)\, dx; \qquad y(0) = 0,\, y(\pi/2) = 3/2.$

   (b). $\displaystyle\int_1^2 \frac{y'^2\, dx}{x^3}; \qquad y(1) = 0,\, y(2) = 15.$

   (c). $\displaystyle\int_0^2 \left(xy' + y'^2\right)\, dx; \qquad y(0) = 1,\, y(2) = 0.$

2. Can light bend along a circular arc, purely through refraction? Explain your answer.

3. Find the geodesics on a right circular cone (as shown in the figure).



4. The Beltrami identity states that the extremal function of the integral

$$I(u) = \int_a^b L(x, u, u')\,dx$$

satisfy the differential equation

$$\frac{d}{dx}\left(L - u'\frac{\partial L}{\partial u'}\right) - \frac{\partial L}{\partial x} = 0.$$

Please prove the identity using the Euler-Lagrange equations and the chain rule. Note that as a special case, when L does not depend on x, we get the equation for the autonomous case, i.e., $H = const$.

5. Newton's aerodynamic problem (the problem of finding the surface of revolution that minimizes drag) is often approximated by assuming the shape is long and thin, so that y' is large (and negative). In this case we can approximate

$$\frac{1}{1+y'^2} \simeq \frac{1}{y'^2}$$

and the functional of interest by

$$F\{y\} \simeq \int_0^R \frac{x}{y'^2}\,dx,$$

Derive the shape that arise from minimizing this functional.