Tutorial 4: Wednesday 12th September

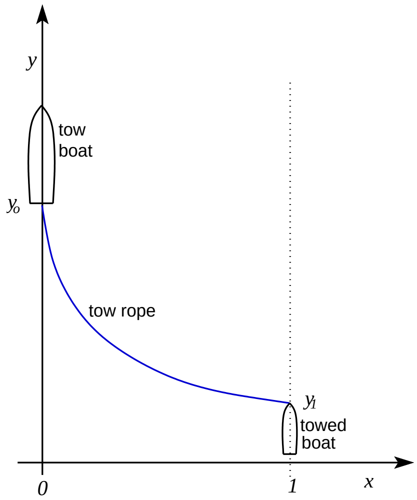
Equality constraints, free end-points and natural boundary conditions.

1. **Non-holonomic constraints:** Consider the catenary problem again. In lectures we rearranged the problem thus
$$W_p\{y\} = \int_0^L m(s) g y(s) ds = mg \int_{x_0}^{x_1} y\sqrt{1+y'^2} dx,$$
However, we assumed that the mass function was constant (i.e., the cable had uniform density). If we use an alternative approach of incorporating a non-holonomic constraint we can do better. Firstly note that if we specify the shape of the curve by $(x(s), y(s))$ then for a small enough triangle
$$\delta x^2 + \delta y^2 = \delta s^2.$$
Dividing by δs^2 and taking $\delta s \to 0$ we get
$$\frac{dx}{ds}^2 + \frac{dy}{ds}^2 = \dot{x}^2 + \dot{y}^2 = 1,$$
which was the starting point for rearranging the functional above. However, here, we will use it as a constraint in the original form of the functional. Use a Lagrange multiplier function $\lambda(s)$ to solve the catenary problem, and the notation that $dy/dx = y'$ and $\dot{x} = dx/ds$, $\dot{y} = dy/ds$.

2. **Isoperimetric constraints:** Consider the catenary problem again. In lectures we solved the problem
$$W_p\{y\} = \int_0^L mgy(s) ds = mg \int_{x_0}^{x_1} y\sqrt{1+y'^2} dx,$$
where the mass of the cable dominated the problem, but now we consider the problem of the design of a suspension bridge. In this case, the cable's weight is negligable compared to the weight of the roadway (or railway) across the bridge, and the cables that join the bridge to the main suspension cable. Here, the problem can be written
$$W_p\{y\} = g \int_0^1 c_1 + c_2 y(x) dx,$$
where c_1 is the density of the bridge platform, and c_2 the average density of the cables between bridge and main cable. Obviously the term c_1 has no affect on the optimization, and we drop this from further consideration.

   If we just consider the function as is, then it is degenerate, and we cannot solve to find the shape of the extremal curves, but when we include the isoperimetric constraint we get a functional of the form
$$W_p\{y\} = \int_0^1 my(x) + \lambda\sqrt{1+y'^2} dx,$$
Solve this to find the shape of the suspension cable in a suspension bridge.

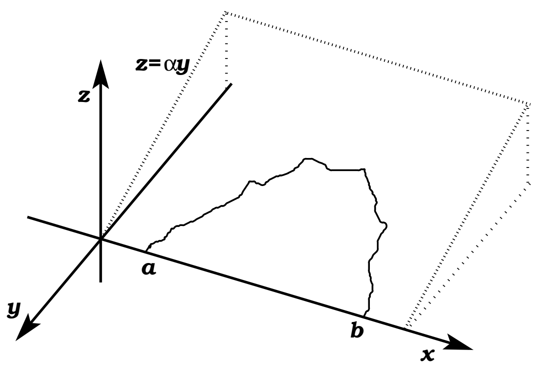
3. **Natural boundary conditions:** Consider the following catenary-like problem. We have two boats (perhaps towing a net between them). However, only the first boat has a motor. The second towed boat is using its rudder to allow it to maintain a constant horizontal distance 1 from the tow boat. See the figure for an illustration:



The tow rope is being pulled through the water, and the force on a section of tow rope is proportional to the length of that section, so its potential energy is similar to that of a hanging cable plus the potential related to the towed boat, which is just Ry_1 where R is the "resistance" of the boat to being pulled through the water. The length of the net is L, and we fix $y(0) = y_0$, but we don't know $y(1) = y_1$.

Derive the shape of tow rope (seen from above), and the natural boundary conditions that apply at the towed boat.

4. **Isoperimetric constraints:**



It is a little known fact that the real-estate laws of ancient Carthage, like those of present-day South Australia, allow a "cooling-off" period before any transaction is finalized. Lucky for Queen Dido that this was true, for no sonner had she figured out that the curve she needed was a **circlular arc**, but one of her slaves discovered that the ground she had purchased was not flat at all but had a (constant) slope. Houses, palaces an so on have to be built on level ground, so how should she select her curve now to get the best deal. i.e., to maximize the enclosed horizontal area?

That is, find the curve $y(x)$ such that

$$F\{y\} = \int_a^b y(x)\, dx$$

is maximized, subject to the length constraint

$$\int_a^b \sqrt{1 + y'^2 + z'^2}\, dx = L$$

for $b - a < L < \pi(b - a)$, and where the curve is constrained to lie in the plane $z = \alpha y$, and the fixed end conditions are

$$y(a) = y(b) = z(a) = z(b) = 0$$

[Note: don't try to solve for the integration constants, or Lagrange multiplier, explain the shape of the curve]