Tutorial 4 Solutions

1. **Non-holonomic constraints:** Consider the catenary problem again. In lectures we rearranged the problem thus

$$W_p\{y\} = \int_0^L m(s) g y(s) ds = mg \int_{x_0}^{x_1} y \sqrt{1 + y'^2} dx,$$

However, we assumed that the mass function was constant (i.e., the cable had uniform density). If we use an alternative approach of incorporating a non-holonomic constraint we can do better. Firstly note that if we specify the shape of the curve by $(x(s), y(s))$ then for a small enough triangle

$$\delta x^2 + \delta y^2 = \delta s^2.$$

Dividing by δs^2 and taking $\delta s \to 0$ we get

$$\frac{dx}{ds}^2 + \frac{dy}{ds}^2 = \dot{x}^2 + \dot{y}^2 = 1,$$

which was the starting point for rearranging the functional above. However, here, we will use it as a constraint in the original form of the functional. Use a Lagrange multiplier function $\lambda(s)$ to solve the catenary problem, and the notation that $dy/dx = y'$ and $\dot{x} = dx/ds$, $\dot{y} = dy/ds$.

Solution:

Incorporating the Lagrange multiplier we get

$$F\{x, y, \lambda\} = \int_0^L \rho(s) y(s) + \lambda(x) \left(\dot{x}^2 + \dot{y}^2 - 1 \right) ds,$$

Note this is a functional of three dependent variables, so we get three E-L equations

$$\begin{aligned} \frac{d}{ds}\frac{\partial f}{\partial \dot{x}} - \frac{\partial f}{\partial x} &= 0 \\ \frac{d}{ds}\frac{\partial f}{\partial \dot{y}} - \frac{\partial f}{\partial y} &= 0 \\ \frac{d}{ds}\frac{\partial f}{\partial \dot{\lambda}} - \frac{\partial f}{\partial \lambda} &= 0 \end{aligned}$$

where $\dot{x} = dx/ds$, $\dot{y} = dy/ds$ and $y' = dx/dy$. which gives us

$$\begin{aligned} \frac{d}{ds} 2\lambda(x)\dot{x} &= 0 \\ \frac{d}{ds} 2\lambda(x)\dot{y} &= \rho(s) \\ \dot{x}^2 + \dot{y}^2 &= 1 \end{aligned}$$

The first equation means that $2\lambda(x)\dot{x} = c_1$ a constant. Now $x' \neq 0$ so

$$\lambda(x) = \frac{c_1}{2\dot{x}}.$$

Substitute λ into $\frac{d}{ds} 2\lambda(x)\dot{y} = \rho(s)$, and we get

$$\begin{aligned} \frac{d}{ds}\frac{\dot{y}}{\dot{x}} &= \frac{\rho(s)}{c_1} \\ \frac{d}{ds}\frac{dy}{ds}\frac{ds}{dx} &= \frac{\rho(s)}{c_1} \\ \frac{d}{ds}\frac{dy}{dx} &= \frac{\rho(s)}{c_1} \\ \frac{d^2y}{dx^2}\frac{dx}{ds} &= \frac{\rho(s)}{c_1} \end{aligned}$$

Note that if the cable density is constant so we scale to take $\rho(s) = 1$, and from our prior study of the catenary we know that $\frac{ds}{dx} = \sqrt{1 + y'^2}$, so the above reduces to

$$c_1 \frac{y''}{\sqrt{1 + y'^2}} = 1$$

which can be integrated with respect to x to get

$$c_1 \sinh^{-1} y' = x - c_2.$$

Inverting we get

$$y' = \sinh\left(\frac{x - c_2}{c_1} \right),$$

which can be integrated again to get

$$y = c_1 \cosh\left(\frac{x - c_2}{c_1} \right).$$

Now we have the density along the cable parameterized by s, but we need to know the density with respect to $\rho(x)x$. The constraint $\dot{x}^2 + \dot{y}^2 = 1$ helps. We can go back to the triangle we used on the original catenary to get

$$ds = \rho(x) \sqrt{1 + y'^2}\, dx.$$

So we get

$$\begin{aligned} \frac{d^2y}{dx^2}\frac{dx}{ds} &= \frac{\rho(s)}{c_1} \\ \frac{d^2y}{dx^2}\frac{dx}{ds} &= \frac{\rho(x)}{c_1}\frac{dx}{ds} \end{aligned}$$

and $\dot{x} > 0$ so

$$y'' = \frac{\rho(x)}{c_1}.$$

If we know the density of the cable with respect to x, this problem is easy to solve by integrating twice. For instance, if the density with respect to x is constant, then the resulting curve is a parabola.

This result is particularly relevant to the design of a suspension bridge where the cable weight is small in comparison to the weight of the road deck, which has constant density with respect x.

2. **Isoperimetric constraints:** Consider the catenary problem again. In lectures we solved the problem

$$W_p\{y\} = \int_0^L mgy(s)ds = mg\int_{x_0}^{x_1} y\sqrt{1+y'^2}dx,$$

where the mass of the cable dominated the problem, but now we consider the problem of the design of a suspension bridge. In this case, the cable's weight is negligable compared to the weight of the roadway (or railway) across the bridge, and the cables that join the bridge to the main suspension cable. Here, the problem can be written

$$W_p\{y\} = g\int_0^1 c_1 + c_2 y(x)dx,$$

where c_1 is the density of the bridge platform, and c_2 the average density of the cables between bridge and main cable. Obviously the term c_1 has no affect on the optimization, and we drop this from further consideration.

If we just consider the function as is, then it is degenerate, and we cannot solve to find the shape of the extremal curves, but when we include the isoperimetric constraint we get a functional of the form

$$W_p\{y\} = \int_0^1 my(x) + \lambda\sqrt{1+y'^2}dx,$$

Solve this to find the shape of the suspension cable in a suspension bridge.

Solution In fact we don't need to solve these, except by recognizing that this is Dido's problem. We can seeking an extremal of a functional describing area under a curve $my(x)$ subject to a distance constraint. The solution is a circular arc as before. However, in this case it should be facing downwards, not upwards.

What's the difference. Consider the Golden Gate bridge. Its main span is 1,280 meters, and the height of the towers above the bridge is 152 meters. The following equations and figure illustrates the optimal catenary, circular arc and parabola. We can see that they are nearly identical at this scale.

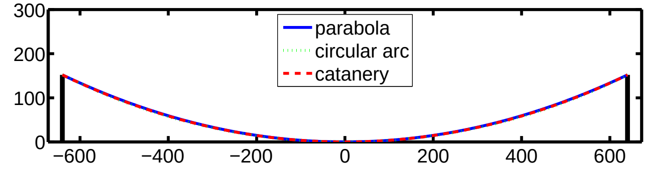
Take the end-towers to be at ± 640 meters, and $y_0 = y_1 = 152$.

$$\begin{aligned} y_{\text{catenary}} &= c_1 \cosh\left(\frac{x}{c_1}\right) - \lambda \\ y_{\text{parabola}} &= Ax^2 + B \\ y_{\text{cirle}} &= y_0 - \sqrt{r^2 - x^2} \end{aligned}$$

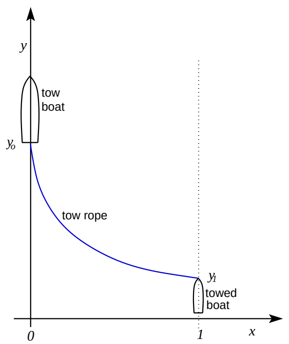
Note that in each case, the middle of the curve is close to the road bed so we want $y(0) \simeq 0$, and I use this constraint rather than the somewhat harder isoperimetric constraint to derive the constants. So

$$\begin{aligned} y_{\text{catenary}} &= c_1\left[\cosh\left(\frac{x}{c_1}\right) - \cosh 0\right] \\ y_{\text{parabola}} &= 152(x/640)^2 \\ y_{\text{cirle}} &= r - \sqrt{r^2 - x^2} \end{aligned}$$

Remaining constants are determined by the end points, e.g., $y(640) = 152$, so that $r = 1423$ and $c_1 = 1372$.



3. Natural boundary conditions: Consider the following catenary-like problem. We have two boats (perhaps towing a net between them). However, only the first boat has a motor. The second towed boat is using its rudder to allow it to maintain a constant horizontal distance 1 from the tow boat. See the figure for an illustration:



The tow rope is being pulled through the water, and the force on a section of tow rope is proportional to the length of that section, so its potential energy is similar to that of a hanging cable plus the potential related to the towed boat, hwich is just Ry_1 where R is the "resistance" of the boat to being pulled through the water. The length of the net is L, and we fix $y(0) = y_0$, but we don't know $y(1) = y_1$.

Derive the shape of tow rope (seen from above), and the natural boundary conditions that apply at the towed boat.

Solutions: Given the form of the forces on the cable, the functional of interest will be the same as that of the catenary plus , i.e.,

$$F\{y\} = Ry_1 + \int_{x_0}^{x_1} (y + \lambda)\sqrt{1 + y'^2} dx,$$

hence the same Euler-Lagrange equations apply, and so the shape will still be a catenary, as we might expect.

$$y = c_1 \cosh\left(\frac{x - c_2}{c_1}\right) - \lambda. \tag{1}$$

However, we cannot solve to find the constants because we don't know y_1 and must derive a natural boundary condition. As $x_1 = 1$ is fixed, the natural boundary condition is

$$\frac{\partial f}{\partial y'} + \frac{\partial \phi}{\partial y} = \left.\frac{(y + \lambda)y'}{\sqrt{1 + y'^2}}\right|_{x=1} + R = 0.$$

Now

$$\begin{aligned} y + \lambda &= c_1 \cosh\left(\frac{x - c_2}{c_1}\right) \\ y' &= \sinh\left(\frac{x - c_2}{c_1}\right). \\ \sqrt{1 + y'^2} &= \cosh\left(\frac{x - c_2}{c_1}\right). \end{aligned}$$

so the condition reduces to

$$c_1 \sinh\left(\frac{x - c_2}{c_1}\right)\bigg|_{x=1} = c_1 \sinh\left(\frac{1 - c_2}{c_1}\right) = -R.$$

Notes: Remember we have two other conditions that must be satisfied

$$\begin{aligned} L\{y\} &= \int_0^1 \sqrt{1 + y'^2}\, dx \\ &= \int_0^1 \cosh\left(\frac{x - c_2}{c_1}\right) dx \\ &= c_1 \left[\sinh\left(\frac{x - c_2}{c_1}\right)\right]_{-1}^{1} \\ &= 2c_1 \left[\sinh\left(\frac{1 - c_2}{c_1}\right) - \sinh\left(\frac{-c_2}{c_1}\right)\right] \end{aligned}$$

and

$$y(0) = y_0.$$

and from these we could (numerically) determine the three constants c_1, c_2 and λ.

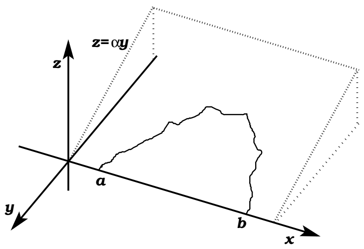
Also note that if $R = 0$, i.e., the towed boat has no resistance, then the condition becomes

$$\sinh\left(\frac{1 - c_2}{c_1}\right) = 0.$$

Now $\sinh$ is only zero at zero, so this condition reverts to saying that $c_1 = 1$, or that the "middle" of the caternary is at the towed boat. At this point $y' = 0$, so if the only resistance comes from the tow rope, then the rope will join the towed boat at right angles.

4. Isoperimetric constraints:

It is a little known fact that the real-estate laws of ancient Carthage, like those of present-day South Australia, allow a “cooling-off” period before any transaction is finalized. Lucky for Queen Dido that this was true, for no sonner had she figured out that the curve she needed was a **circular arc**, but one of her slaves discovered that the ground she had purchased was not flat at all but had a (constant) slope. Houses, palaces an so on have to be built on level ground, so how should she select her curve now to get the best deal. i.e., to maximize the enclosed horizontal area?



That is, find the curve $y(x)$ such that

$$F\{y\} = \int_a^b y(x)\, dx$$

is maximized, subject to the length constraint

$$\int_a^b \sqrt{1 + y'^2 + z'^2}\, dx = L$$

for $b-a < L < \pi(b-a)$, and where the curve is constrained to lie in the plane $z = \alpha y$, and the fixed end conditions are

$$y(a) = y(b) = z(a) = z(b) = 0$$

[Note: don’t try to solve for the integration constants, or Lagrange multiplier, explain the shape of the curve]

Solution:

Given the problem has constraints, we should include Lagrange multipliers λ, so that we look for extremals of

$$H\{y\} = \int_a^b y(x) + \lambda\sqrt{1 + y'^2 + z'^2}\, dx = \int_a^b y(x) + \lambda\sqrt{1 + y'^2(1+\alpha^2)}\, dx$$

given the constraint that $z = \alpha y$. The function $h(y, y')$ does not depend explicitly on x, so we can form the function $H(y, y')$ [Hint: Don’t confuse the function $H(y, y')$ with the functional $H\{y\}$] such that

$$\begin{aligned}
H(y,y') = y'\frac{\partial h}{\partial y'} - f &= \frac{\lambda(1+\alpha^2)y'^2}{\sqrt{1+y'^2(1+\alpha^2)}} - y - \lambda\sqrt{1+y'^2(1+\alpha^2)} = c_1 = const \\
\Rightarrow \quad & \frac{\lambda(1+\alpha^2)y'^2 - \lambda - \lambda y'^2(1+\alpha^2)}{\sqrt{1+y'^2(1+\alpha^2)}} - y = c_1 \\
\Rightarrow \quad & \frac{-\lambda}{\sqrt{1+y'^2(1+\alpha^2)}} = y + c_1 \\
\Rightarrow \quad & \lambda^2 = (y+c_1)^2(1+ \\
\Rightarrow \quad & y'^2(1+\alpha^2)(y+c_1)^2 = \lambda^2 - (y+c_1)^2 \\
\Rightarrow \quad & y'^2 = \frac{\lambda^2 - (y+c_1}{(1+\alpha^2)(y+} \\
\Rightarrow \quad & \frac{dy}{dx} = \frac{\sqrt{\lambda^2 - (y+}}{\sqrt{(1+\alpha^2)(y}} \\
\Rightarrow \quad & dx = \frac{\sqrt{(1+\alpha^2)(y}}{\sqrt{\lambda^2 - (y+}}
\end{aligned}$$

Integrating, we use two changes of variables, to get

$$\begin{aligned}
x + c_2 &= \int \frac{\sqrt{(1+\alpha^2)}(y+c_1)}{\sqrt{\lambda^2 - (y+c_1)^2}}\, dy \\
&= \sqrt{(1+\alpha^2)} \int \frac{w}{\sqrt{\lambda^2 - w^2}}\, dw \\
&= \sqrt{(1+\alpha^2)} \int \frac{1/2}{\sqrt{\lambda^2 - s}}\, ds \\
&= \sqrt{(1+\alpha^2)}\sqrt{\lambda^2 - s} \\
&= \sqrt{(1+\alpha^2)}\sqrt{\lambda^2 - (y+c_1)^2}
\end{aligned}$$

Squaring both sides, and rearranging we get

$$(x + c_2)^2 + (1+\alpha^2)(y+c_1)^2 = \lambda^2$$

which is the equation of an **ellipse**. Notice that this ellipse is just the projection of a circular arc on the surface $z = \alpha y$ onto the plane $z = 0$. So, Dido is safe, just get her servants to draw up a circular arc on the sloped surface, and it will automatically encompass the largest possible horizontal area (but this area will then be elliptical, and somewhat smaller than the sloped region encompasses).