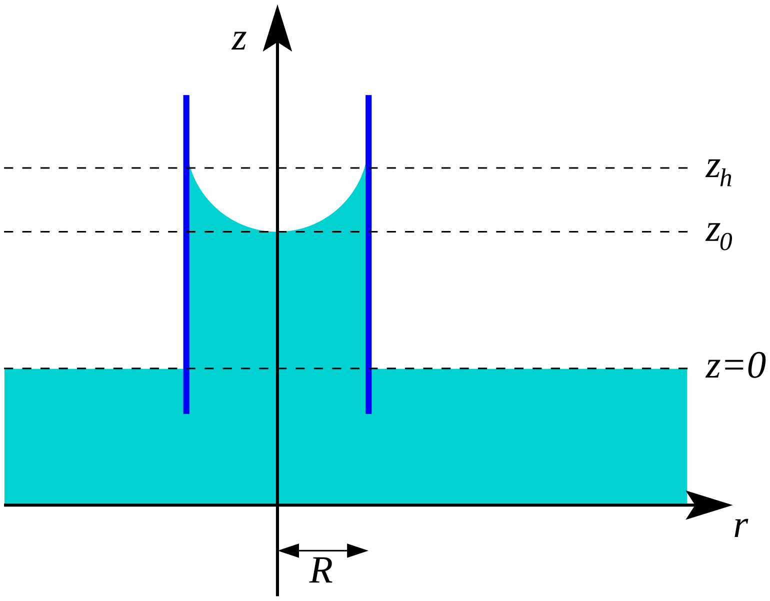
Tutorial 5: Wednesday 10th October

1. **Capillaries:** Imagine a slender, open-end cylinder dipped into a large water bath. It is well known that the cylinder acts as a capillary tube, and the water will rise up the tube, and moreover that the shape of the surface of the water inside the tube will have a curved shape, as shown in the figure.



For convenience, we assume the water level outside the cylinder is at height $z = 0$, and that the cylinder's center of rotation is the z-axis. We will consider it in cylindrical coordinates (r, θ, z), where the cylinder will have radius R. Given the radial symmetry of the problem, we will describe the height of water in the cylinder by $z(r)$, and we denote $z_0 = z(0)$ and $z_h = z(R)$, but note that these are not fixed boundary conditions (the end-points are free).

At equilibrium, the potential energy of this system will be minimized. The potential energy is made up of the following components:

- The gravitational potential:

$$G\{z\} = 2\pi \Delta\rho g \int_0^R r \int_0^z s \, ds \, dr = 2\pi \Delta\rho g \int_0^R r \frac{z^2}{2} \, dr,$$

where $\Delta\rho$ is the difference between the density of the liquid and air, and g is the gravitational constant.

- The surface energy in the interfaces between the liquid and solid (the cylinder walls), the gas and solid, and the air and liquid, given by

$$\begin{aligned} S\{z\} &= \Delta\gamma \Delta A(\Omega_{SL}) + \gamma_{LG} \Delta A(\Omega_{LG}) \\ &= 2\pi R \Delta\gamma z_h + 2\pi \gamma_{LG} \int_0^R r\sqrt{1 + z'^2} - r \, dr, \end{aligned}$$

where

- The constant parameters γ_{SG}, γ_{SL} and γ_{LG} are the respective parameters determining the strength of affiliation or attraction between these three components. We can think of a parameter γ as the tension on the surface with units corresponding to force along a line of unit length. We take $\Delta\gamma = \gamma_{SL} - \gamma_{SG}$.
- Ω_{SG}, Ω_{SL} and Ω_{LG} are the surfaces between the respective phases.
- $A(\cdot)$ denotes the surface area. $\Delta A(\cdot)$ represents the surface area deformation from the case without the cylinder, for instance, the undeformed surface Ω_{LG} would have area given by a circular disk (radius R, area $\pi R^2 = 2\pi \int_0^R r\, dr$), and the undeformed surface Ω_{LS} would correspond to the liquid in the cylinder at the same height as the water bath.

so the integrals above are trying to minimize the energy resulting from tension in the surfaces due to their deformation from the case without the cylinder. The first integral is the energy in the gas-liquid surface, and the second is the energy resulting in the liquid-solid surface minus the energy from the solid-gas interface it replaces.

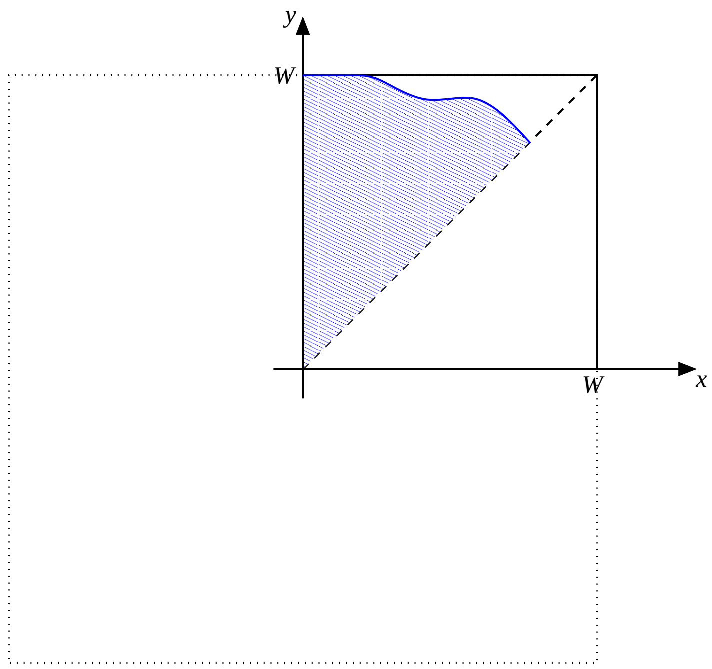
Use calculus of variations to determine the height and shape of the water surface inside the cylinder.

2. **Inequality constraints and broken extremals:** solve the isoperimetric problem inside a square region, i.e., what is the shape that contains the largest area without exceeding a given perimeter L, given that the shape must be entirely contained in a square with sides $2W$ in length.

   Note that the problem is uninteresting for $W > L/2\pi$ because a circle of radius $R = L/2\pi$ satisfies the isoperimetric constraint, and fits inside the square, and by previous work this is clearly the maximal area region (though there are actually multiple possible circles that might fit). Likewise, $8W < L$ is uninteresting, because we cannot meet the perimeter constraint without having a concave shape, so the obvious solution is to contain the entire area of the square, but have the perimeter dip into the shape along a line enclosing zero area. So we consider the case

$$L/8 < W < L/2\pi.$$

   We will simplify the problem in a few ways. Firstly, the reflective symmetries of the problem suggest that we could consider $1/8$ of the square, rather than the whole square (see the figure, which shows $1/8$ of the square with sides $2W$.).



So the inequality constraints on the problem become:

$$\begin{aligned} y(x) &\leq W \\ y(x) &\geq x \end{aligned}$$

where the left end point may move along the y-axis (between the constraints), and the right end-point is free to move along the boundary $x = y$. We will define the end-point to be (x_1, y_1).

The area and perimeter of the region are easily measured by

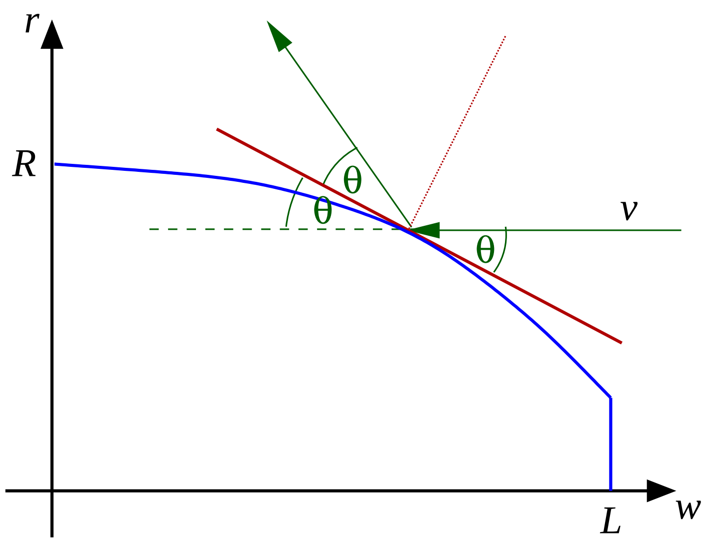
$$A\{y\} = 8 \int_0^{x_1} y - x \, dx. \quad \text{and} \quad L\{y\} = 8 \int_0^{x_1} \sqrt{1 + y'^2} \, dx.$$

Ignoring the factor of 8 in each term, and including the isoperimetric constraint into the problem via a Lagrange multiplier, we obtain an objective function

$$J\{y\} = \int_0^{x_1} y - x + \lambda \sqrt{1 + y'^2} \, dx.$$

Find the shape that maximizes the area without exceeding the perimeter constraint.

3 Terminal costs and optimal control: We originally posed Newton's aerodynamic nose-cone problem for a nose cone pointed upwards (with flow downwards). We could equally have posed it with the flow from right to left as in the figure below, where the shape is described by $r(u)$, u being the horizontal axis.



In this case, a similar simplification of the problem reduces the function of interest to

$$\frac{1}{2\pi}F\{r\} = \frac{1}{2}r(L)^2 + \int_0^L \frac{rr'^3}{1+r'^2}\,dw$$

Questions:

- Show that the above function results from a simple transformation of the previous problem.
- Use natural end-point conditions for a problem with a terminal cost to determine an equation to find $r(L)$.