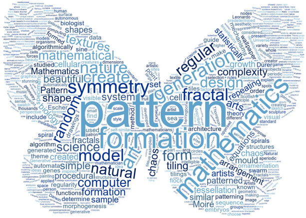
Advanced Mathematical Perspectives 1

Lecture 21: Fractals and Statistical Self-Similarity



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www.maths.adelaide.edu.au/matthew.roughan/notes/AMP1/

School of Mathematical Sciences,
University of Adelaide

THE UNIVERSITY of ADELAIDE

SUB CRUCE LUMEN

ACEMS

AUSTRALIAN RESEARCH COUNCIL CENTRE OF EXCELLENCE FOR MATHEMATICAL AND STATISTICAL FRONTIERS

So, Nat'ralists observe, a flea
Hath smaller fleas that on him prey;
And these have smaller still to bite 'em
And so proceed ad infinitum

Jonathon Swift, 1733

Why is geometry often described as "cold" and "dry?" One reason lies in its inability to describe the shape of a cloud, a mountain, a coastline, or a tree. Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line. Nature exhibits not simply a higher degree but an altogether different level of complexity.

Benoit Mandelbrot

Section 1

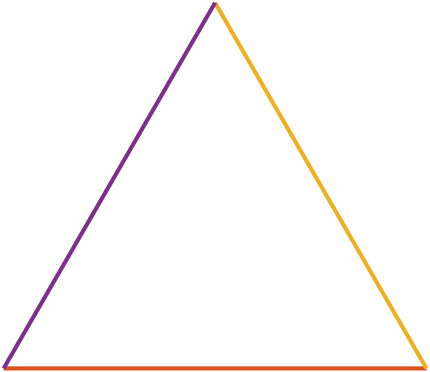
Fractals

The first (deterministic) fractal (1883)

The Cantor set

- Built by repeating a rule, so that the result looks the same under scaling transformations, *i.e.*, it has a symmetry called scale invariance or *self-similarity*
- But Cantor didn't call it a fractal

Self-similarity: von Koch Snowflake (1904)



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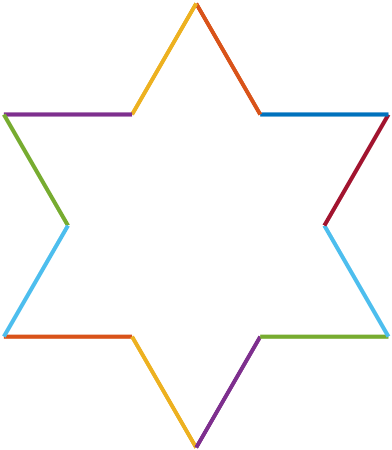

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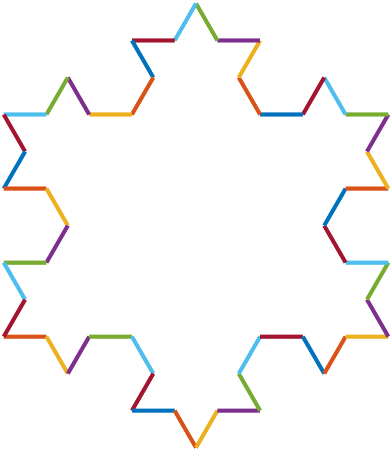

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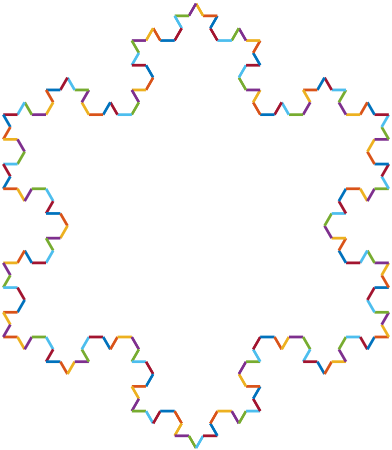

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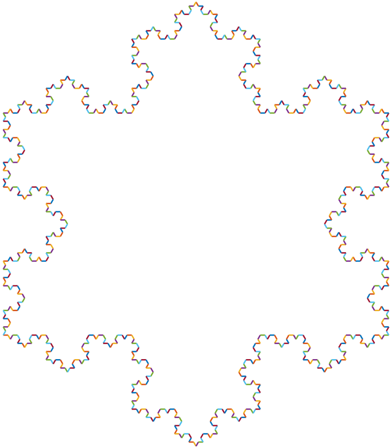

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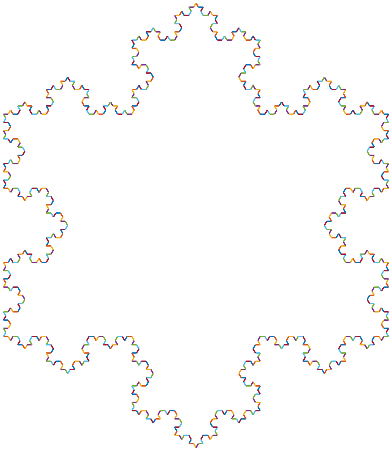
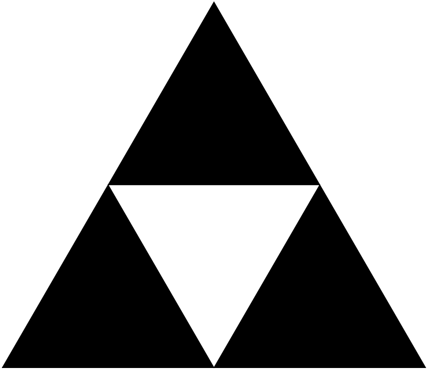


And on *ad infinitum*

Sierpinski Gasket or Triangle

The Sierpinski Gasket appeared c13th century, long before was "discovered" [CT11]



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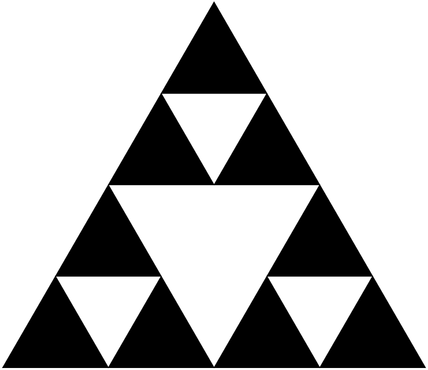

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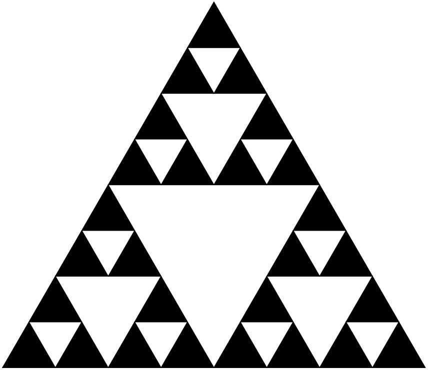

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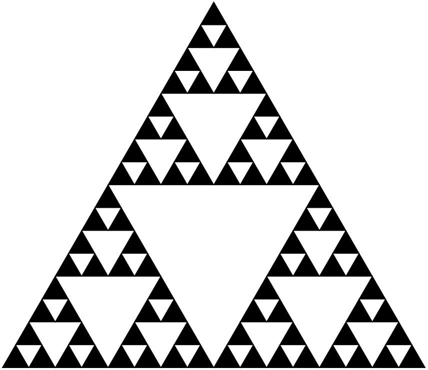

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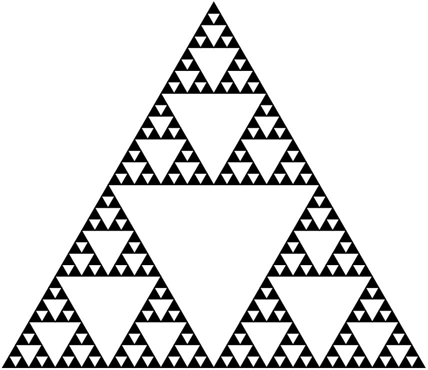

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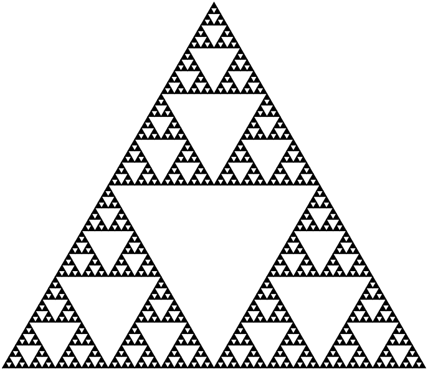
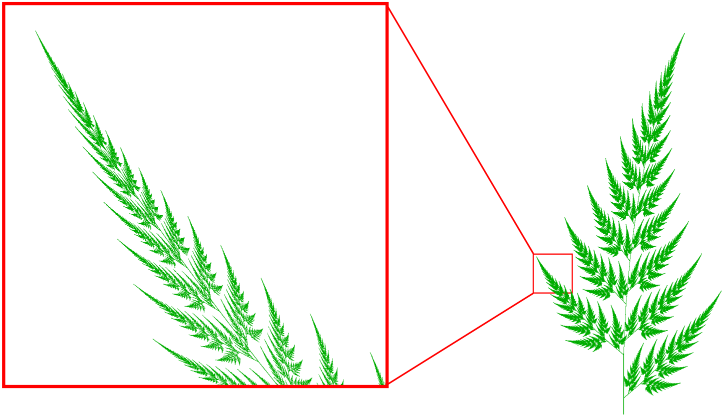


And on *ad infinitum*

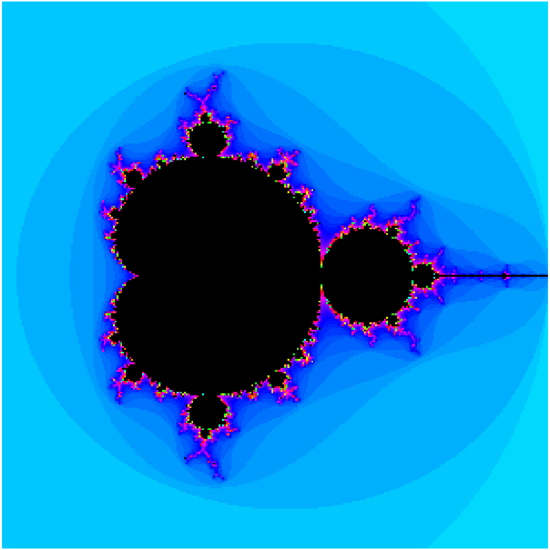
Self-similarity: Iterated Function System (IFS) Fern



C code from
http://astronomy.swin.edu.au/~pbourke/fractals/
But the link is dead, so guess you should head here:
http://paulbourke.net/fractals/

Mandelbrot Set

Fractals aren't defined in terms of strict self-similarity



http://aleph0.clarku.edu/~djoyce/julia/julia.html

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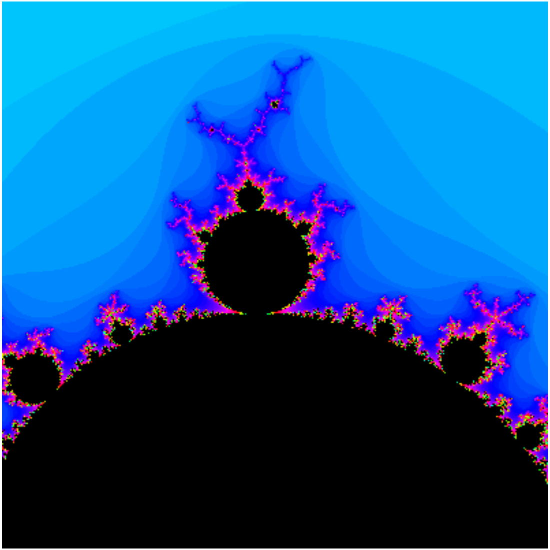

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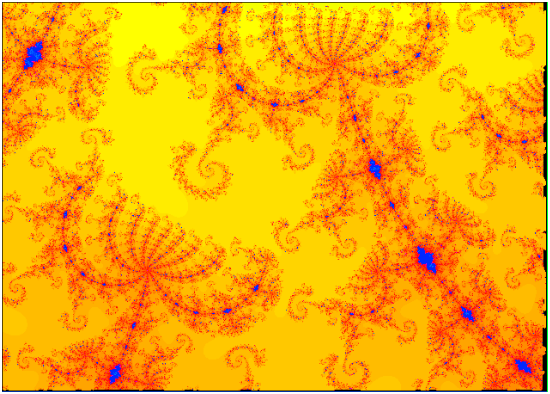


http://www.softsource.com/softsource/fractal.html

Mandelbrot Set DIY

Take the set of functions: $f_c : \mathbb{C} \to \mathbb{C}$ for $c \in \mathbb{C}$, defined by

$$f_c(z) = z^2 + c$$

The **Mandelbrot set** is

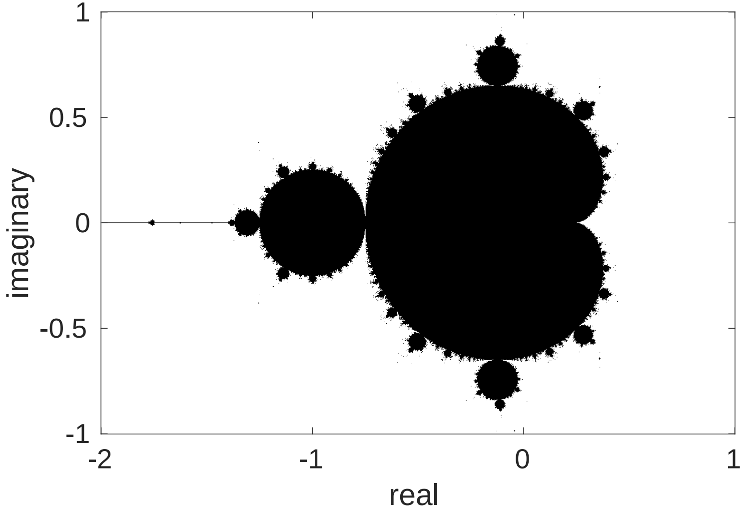
$$M = \left\{ c \in \mathbb{C} \,\middle|\, \sup_{n \in \mathbb{N}} |f_c^n(0)| < \infty \right\}$$

- starting point $z_0 = 0$
- for each point c in the complex plane, iterate $f_c(\cdot)$
- if it diverges, it's not in the set

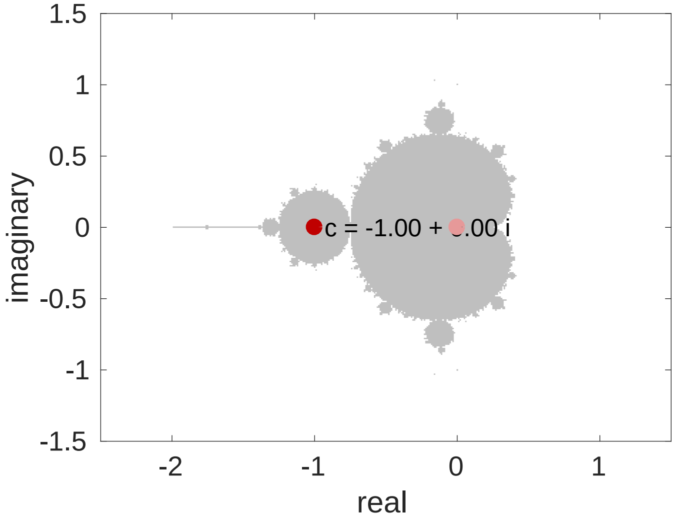
That defines the "set", but usually people plot colours on the points depending on how quickly they diverge.

Mandelbrot "Set"

Axis refer to real and imaginary parts of c, black points are in the set.



Mandelbrot Set Example Sequence I



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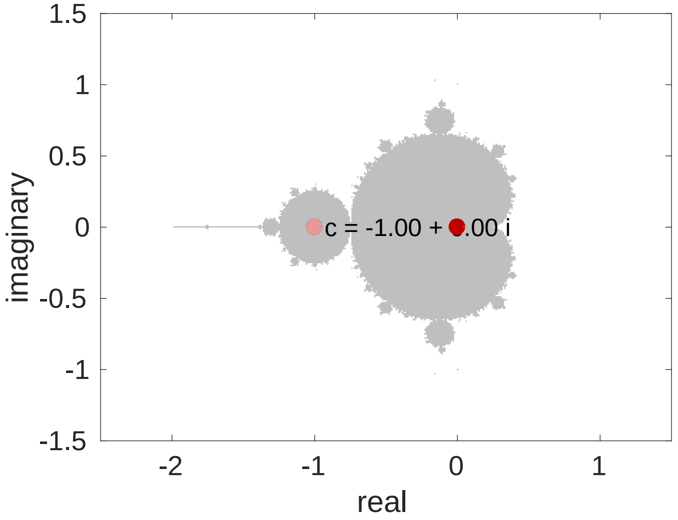

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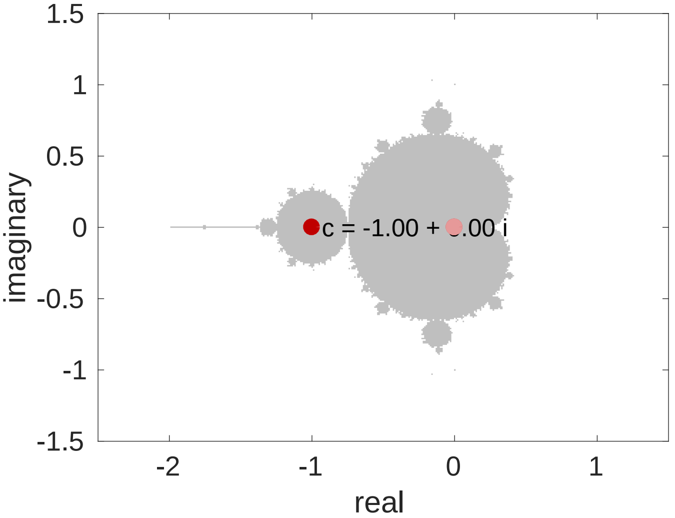

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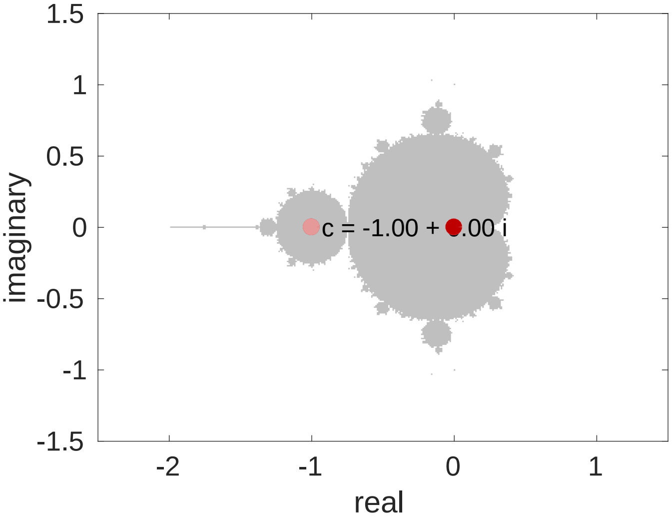
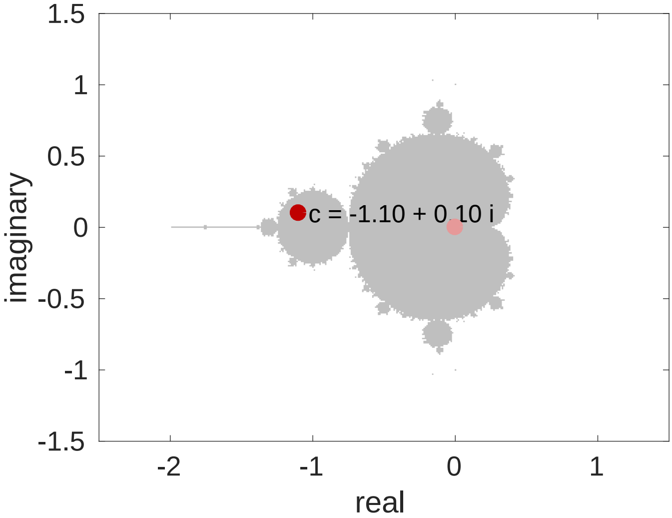


Mandelbrot Set Example Sequence II



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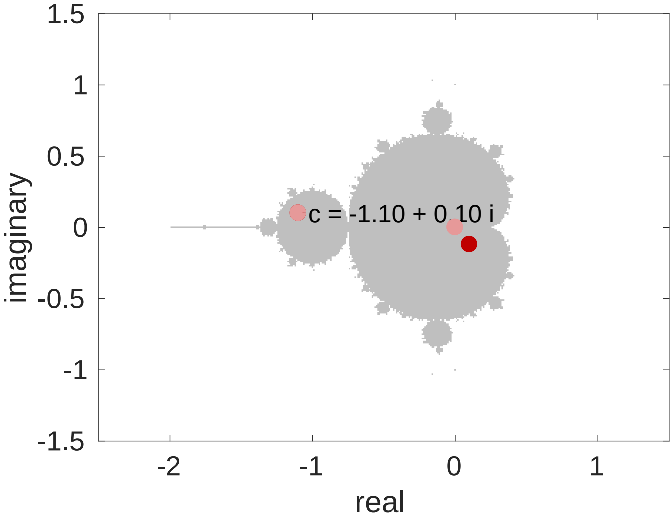

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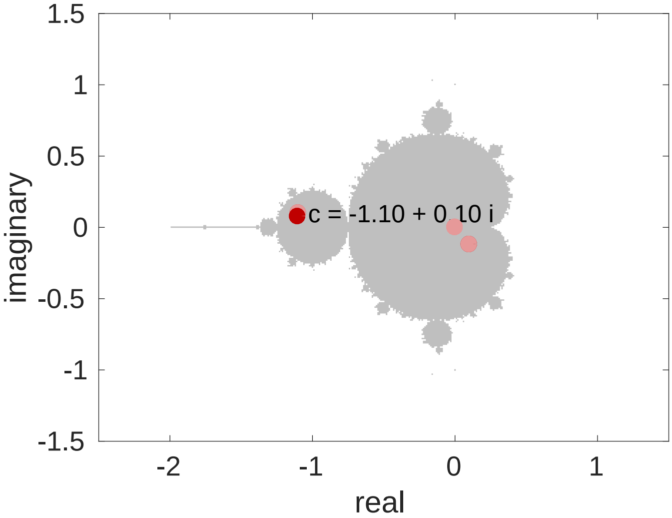

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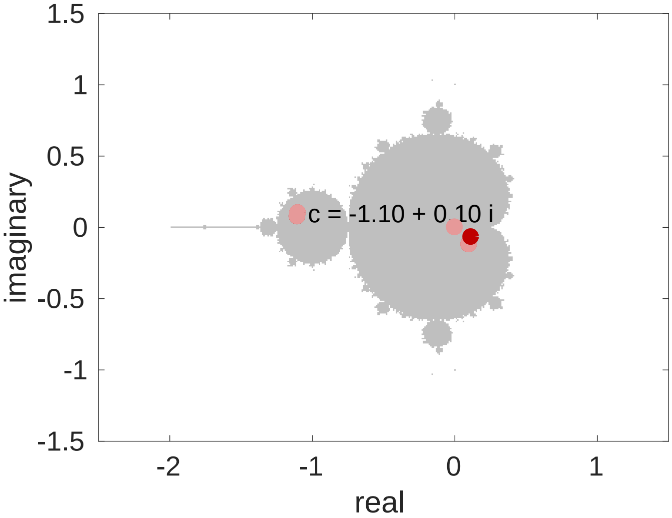

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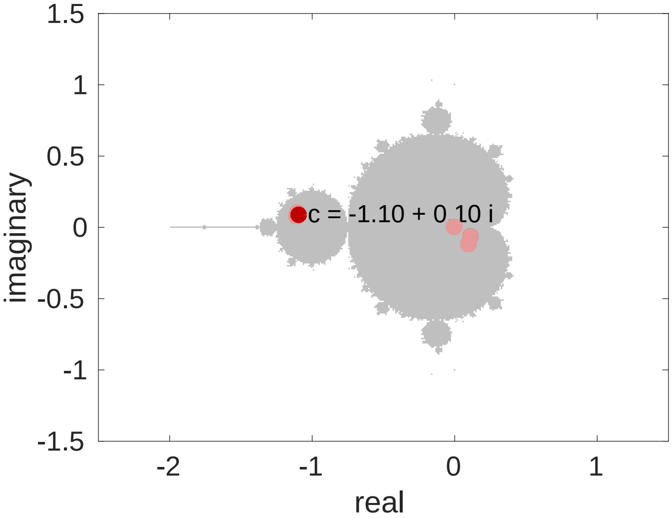

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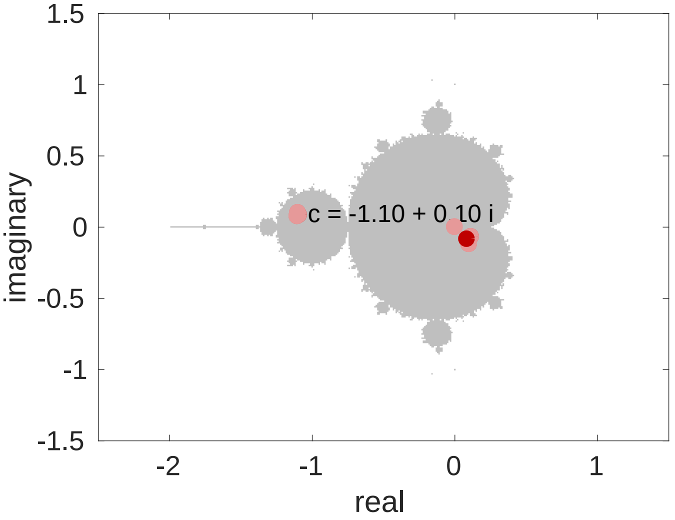

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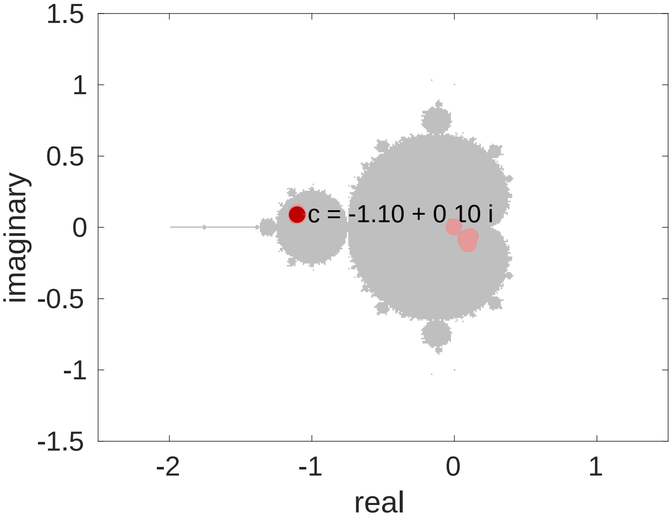

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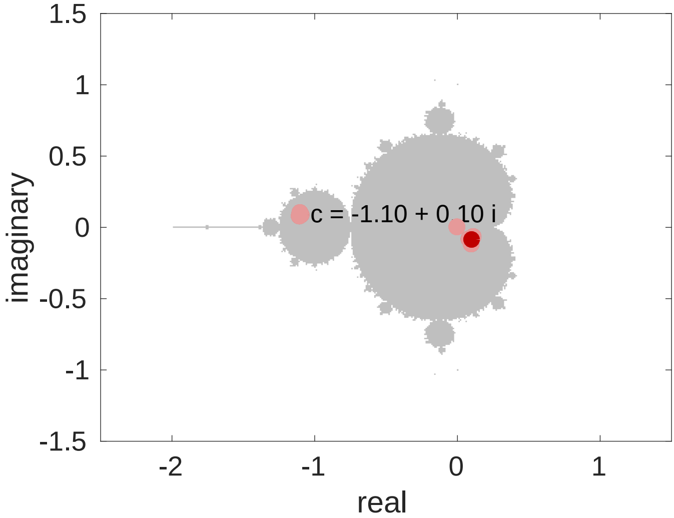

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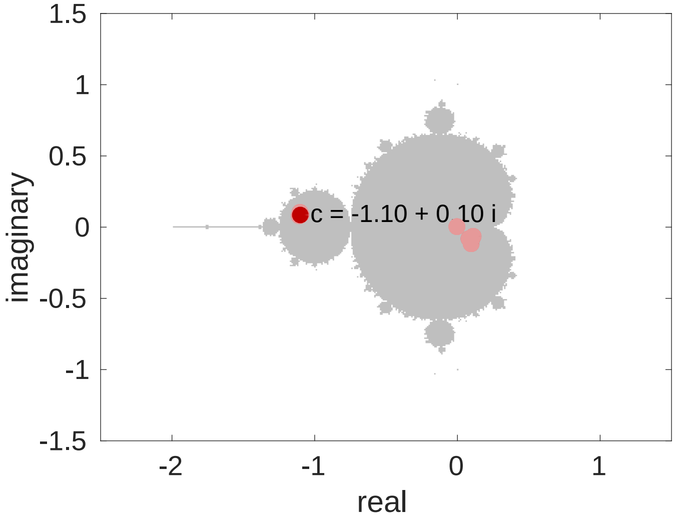

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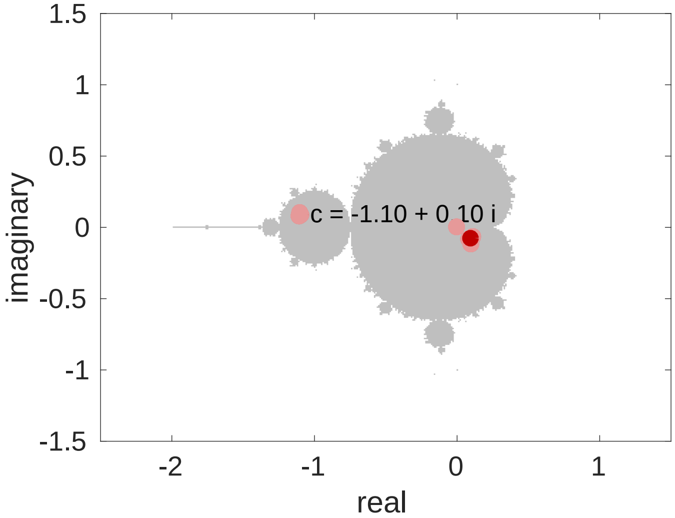
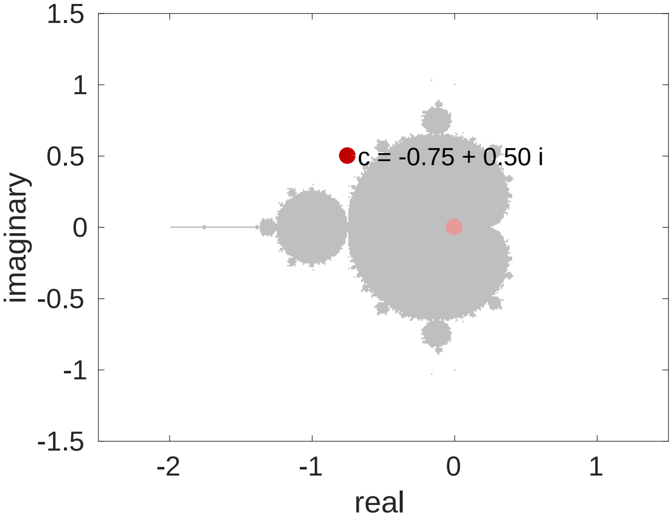
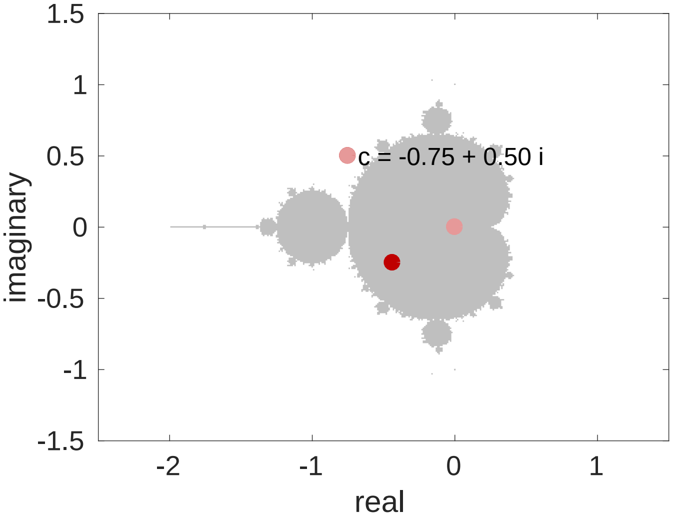


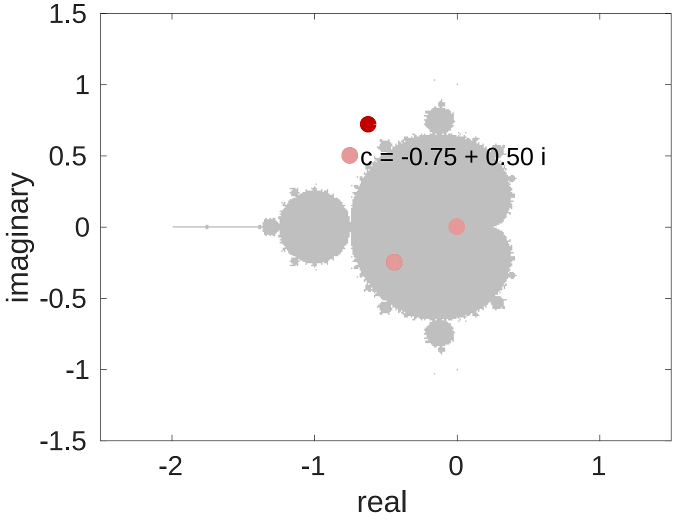
Mandelbrot Set Example Sequence III



Mandelbrot Set Example Sequence III



Mandelbrot Set Example Sequence III



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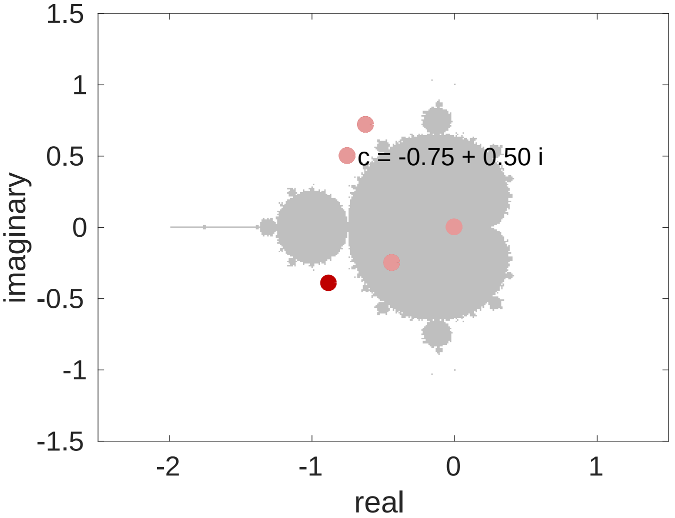

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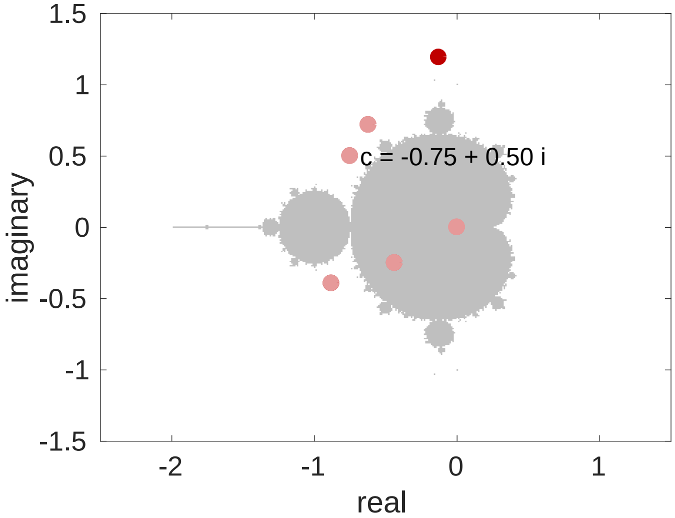

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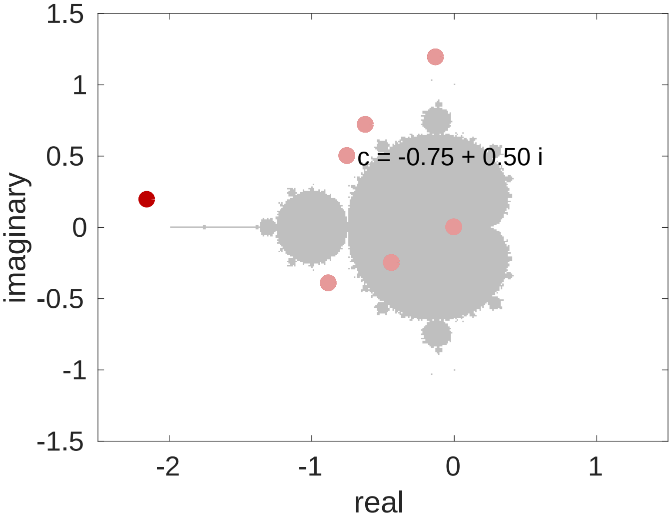
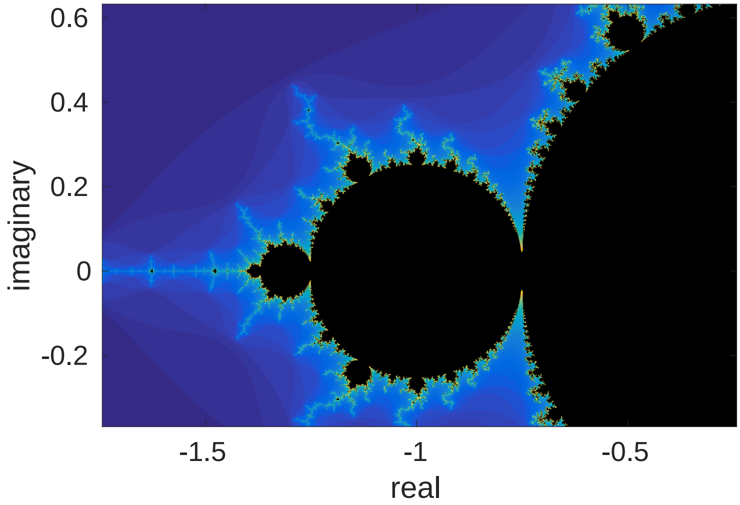


Mandelbrot Set: zooming in



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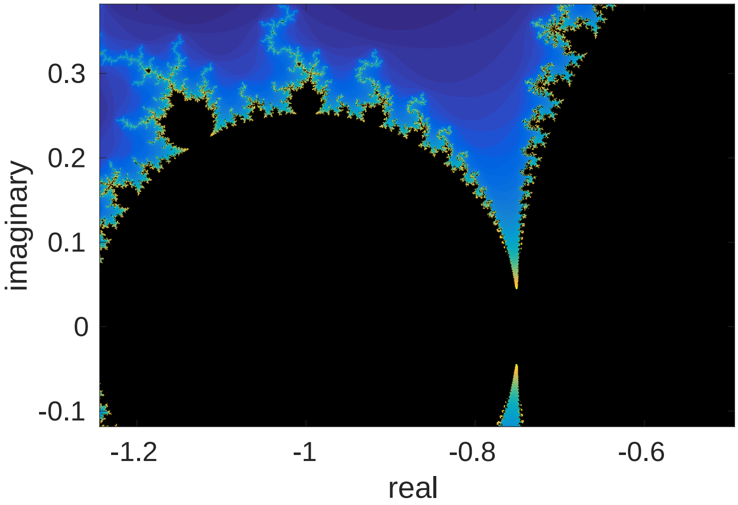

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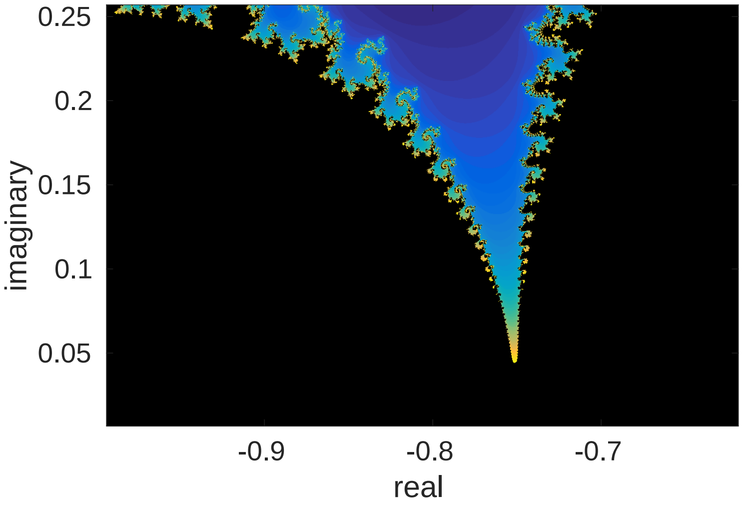

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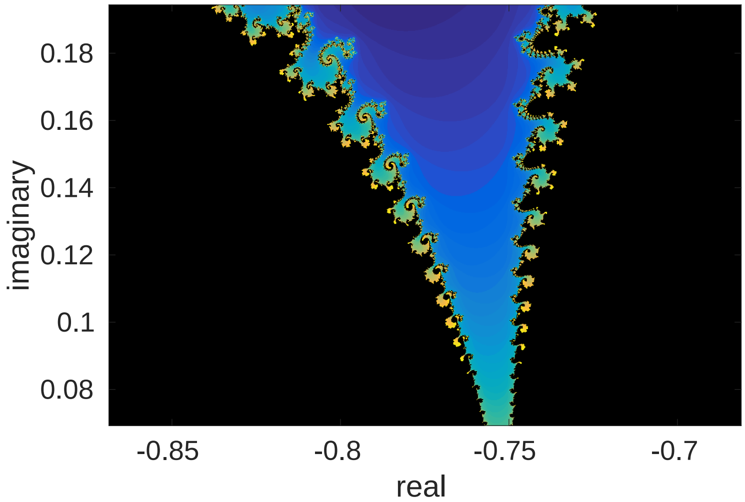

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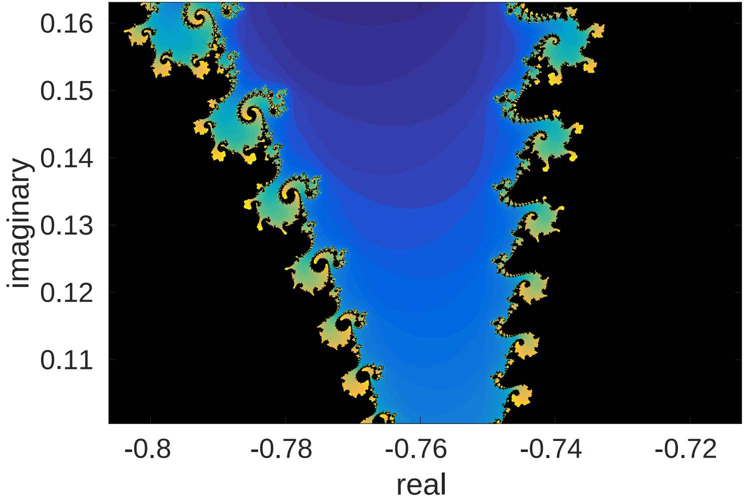

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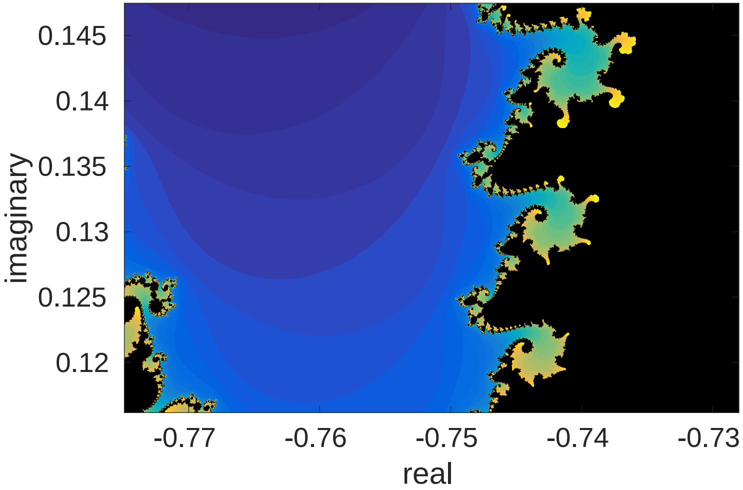

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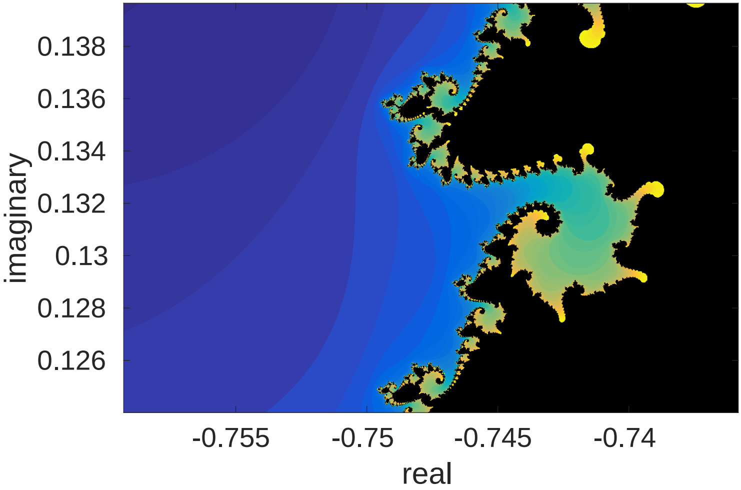

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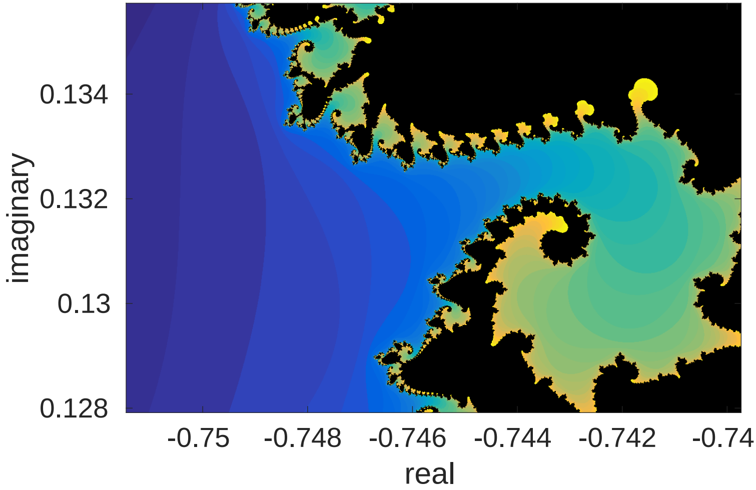

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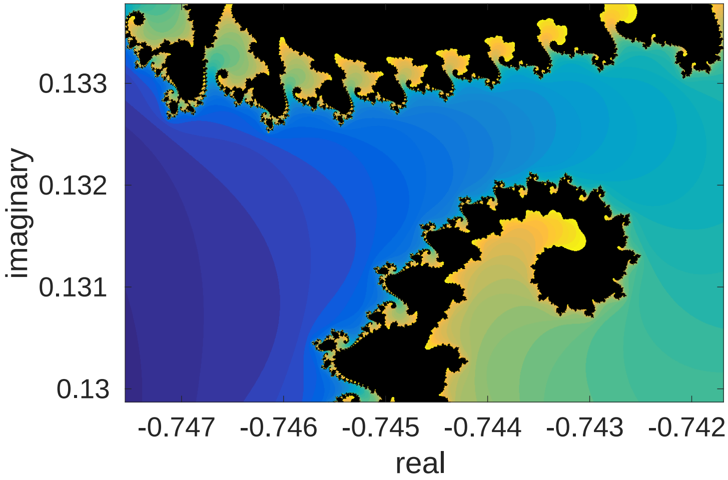

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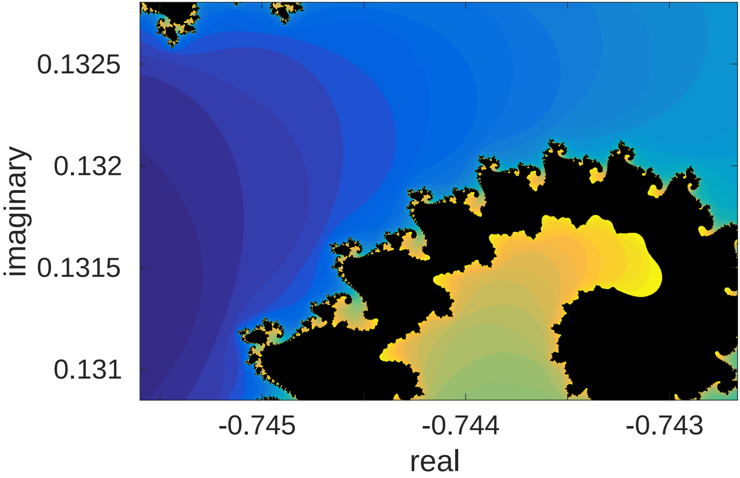

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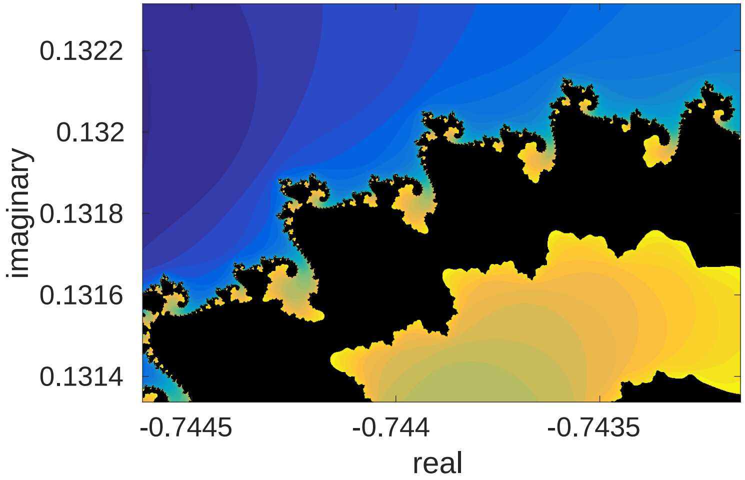


Much more to learn

There is MUCH more to learn about fractals

- Fractional dimensions
- The relationship with *Chaos* and *strange attractors*
- Fractal compression
- Space filling curves
- L-systems
- More on IFSs

Fractals and Art

> I wonder whether fractal images are not touching the very structure of our brains. Is there a clue in the infinitely regressing character of such images that illuminates our perception of art? Could it be that a fractal image is of such extraordinary richness, that it is bound to resonate with our neuronal circuits and stimulate the pleasure I infer we all feel?
>
> *P. W. Atkins*

- Balance between complexity and structure, predictability and randomness, is appealing to us
- Fractal patterns are aethetically pleasing and stress-reducing
  Stress reduction from looking at fractals has been measured
- Fractal analysis has been used to separate fakes

Fractals and Art

"Accidental" examples

- Leonardo da Vinci's Turbulence (c1500)
- Hokusai's Great Wave off Kanagawa (1826)
- Jackson Pollock's Blue Pole's (1952)
  Sometimes called "fractal expressionism"
- Architecture
  - https://en.wikipedia.org/wiki/Virupaksha_Temple,_Hampi
- Data-driven art
- And almost any picture of a natural scene ...

Fractals and Art

Deliberate examples (https://en.wikipedia.org/wiki/Fractal_art) in Pictures

- https://www.creativebloq.com/computer-arts/5-eye-popping-examples-fractal-art-71412376
- https://www.smashingmagazine.com/2008/10/50-phenomenal-fractal-art-pictures/
- https://en.wikipedia.org/wiki/The_Beauty_of_Fractals
- Kerry Mitchell http://www.kerrymitchellart.com/ [Mit] https://www.fractalus.com/info/manifesto.htm and https://en.wikipedia.org/wiki/Kerry_Mitchell

Fractals and Art

Deliberate examples (https://en.wikipedia.org/wiki/Fractal_art) elsewhere

- Movies
  - Fractals in Guardians of the Galaxy Vol 2.
  - Dr Strange (also back to comics of Steve Ditko)
- Music
  - Bruno Degazio http://www-acad.sheridanc.on.ca/~degazio/AboutMeFolder/MusicPages/musiccomp.html [Deg86]
  - https://quod.lib.umich.edu/s/spobooks/bbv9810.0001.001
- Games
  - Fractal's used in procedural generation software, *e.g.*, Terragen http://planetside.co.uk/whats-new-in-terragen-4/
  - more examples https://en.wikipedia.org/wiki/Scenery_generator

Fractals in Inkscape

- Inkscape has several fractal-generation tools
  - L-systems
    http://people.cornellcollege.edu/dsherman/inkscape-fractal.html
    https://thebrickinthesky.wordpress.com/2013/03/17/l-systems-and-penrose-p3-in-inkscape/
    http://tavmjong.free.fr/INKSCAPE/MANUAL/html/Paths-LivePathEffects-VonKoch.html
    - von Koch snowflakes
    - Sierpinkski gaskets
    - ...
  - Fractalize extension
    http://prosepoetrycode.potterpcs.net/2015/08/fractal-rivers-with-inkscape/
    More on this now.

Section 2

Statistical Self-Similarity

Fractals and Modelling

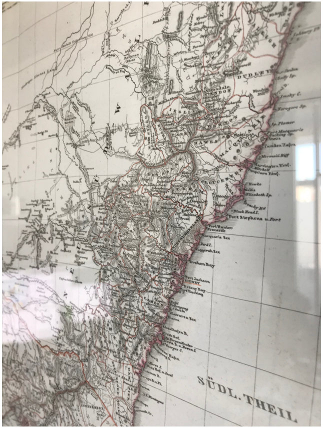
- Fractals (mostly) are deterministic
  - sometimes they look complex enough to be random
  - but the "model" is still deterministic
- Many observed phenomena have similar characteristics but are not at all regular
  - Geographic features: *e.g.*, mountains, rivers, coastlines
  - Biological systems: *e.g.*, blood vessels
  - Time series: rainfall patterns, cardiac rhythms, ...
  - And more: *e.g.*, clouds, snowflakes, ...

  How do we model these?

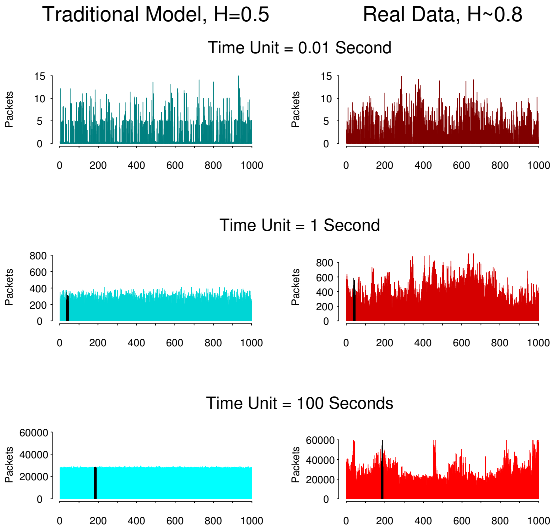
Statistical Self-Similarity Application: Coastline

Coastline paradox: the length of the coastline depends on the length of the rule you use to measure it [Ric61, Man67].

Known as the Richardson effect.



Statistical Self-Similarity Application: Internet Traffic



Statistical Self-Similarity

A "statistic" of a "process" or *time series* X_t is a number that we derive as a summary or measurement of that process

- Mean or average

$$\mu_X = \mathbb{E}[X]$$

- Variance (similarly the standard deviation)

$$\sigma_X^2 = \mathrm{Var}(X)$$

- Lots of others
  - autocovariance
  - spectrum (from Fourier transform)

Statistical Self-Similarity

First, shift the time series so that it has mean 0, *i.e.*, subtract μ_X.

- we often do this as a first step in analysis, because the long-term average is the first thing we look at, and then we analyse whatever was left.

Now imagine forming a *aggregated* time series $\{X_k^{(m)}\}$ at level m by grouping the data into blocks of length m, and averaging

$$X_k^{(m)} := \frac{X_{(k-1)\,m+1} + \cdots + X_{k\,m}}{m}.$$

Linearity of the expectation operator means that

$$\mathbb{E}\left[X_k^{(m)}\right] = 0$$

But what about the other statistics?

Statistical Self-Similarity

If, the process satisfies a scaling relationship such that

$$m^{1-H} X^{(m)}$$

has the same statistics (*e.g.*, variance) as the original $X^{(1)}$ then we say the process is *statistically self-similar* with Hurst parameter H.

Statistical Self-Similarity Example

The $1/f^\alpha$ noise we looked at before with spectrum

$$f_x(s) \sim c_f |s|^{-\alpha}, |s| \to 0$$

is statistically self-similar with Hurst parameter $\alpha = 2H - 1$ or

$$H = \frac{\alpha + 1}{2}$$

- the case $H = 0.5$ corresponds to the central limit case, or $\alpha = 0$ or "white" noise
- but can have $0 < H \leq 1$ as well; for $H > 0.5$ we get "pink" noise, ...

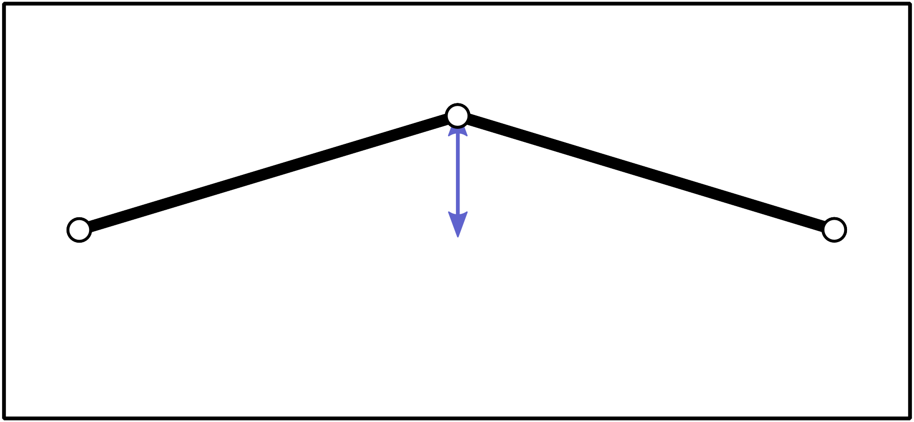
Mid-point displacement algorithm [FFC82]

- Take $H \in [1,2]$, and a variance σ, and define two end points.
- Displace the mid-point by a random amount with mean 0 and variance σ^2
- Repeat this for each of the resulting line segments, but scale the variance by $1/2^{2H}$
- Continue this process recursively

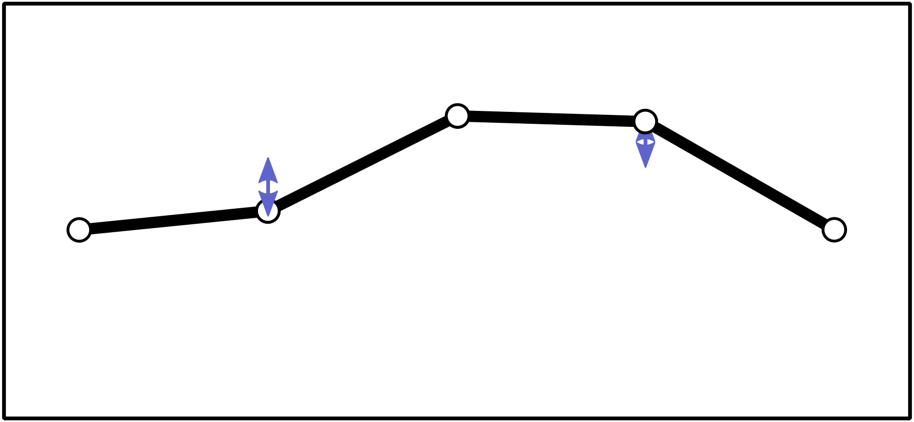
http://old.cescg.org/CESCG97/marak/node3.html

Mid-point displacement algorithm example

Mid-point displacement algorithm example

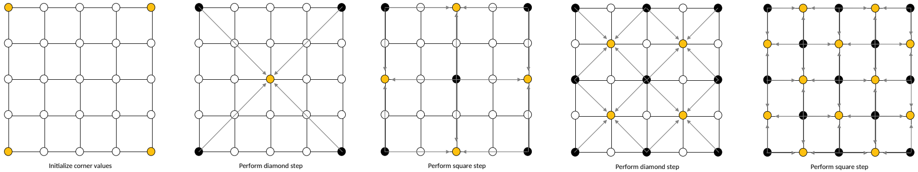


Mid-point displacement algorithm example



Mid-point displacement algorithm in 2D [FFC82]

Sometimes called the diamond-square algorithm



https://en.wikipedia.org/wiki/Diamond-square_algorithm

Used to generate "terrain"
http://planetside.co.uk/free-downloads/terragen-4-free-download/

Mid-point displacement algorithm

Variants

- Move location of mid-point in x and y directions
- Make boundaries periodic
- Higher dimensions (3D = clouds?)
- Approximate an underlying shape, with some smoothness
- Add colours
- Add dynamics (crumbled paper)

Takeaways

- Fractals: complexity and non-locality from simple, local rules
- Statistical Self-Similarity: empirical modelling of real self-similar data
- Fractals are a empirical model, *i.e.*, they model phenomena we see, but often not an explanatory model, *i.e.*, they don't explain why we see what we see
  - but you can go deeper here, it's just beyond the scope of this course

Section 3

Extras

Further reading I

Elisa Conversano and Laura L. Tedeschini, *Sierpinsky triangles in stone, on medieval floors in Rome*, Journal of Applied Mathematics **4** (2011), no. 4.

Bruno Degazio, *Musical aspects of fractal geometry*, International Computer Music Conference, 1986.

Alain Fournier, Don Fussell, and Loren Carpenter, *Computer rendering of stochastic models*, Communications of the ACM **25** (1982), no. 6, 371–384.

Benoit Mandelbrot, *How long is the coast of Britain? statistical self-similarity and fractional dimension*, Science **156** (1967), no. 3775, 636–638.

L. Kerry Mitchell, *Techniques for artistically rendering space-filling curves*.

Lewis F. Richardson, *The problem of contiguity: An appendix to statistics of deadly quarrels*, Yearbook of the Society for the Advancement of General Systems Theory (1961), 139187.