Communications Network Design
Class Exercise 2 Solutions

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1. 5 marks The cost function is $C(\mathbf{f}) = \sum_{e \in L} \alpha_e f_e + \beta_e$.

Start with direct routing. The cost of the network are shown in Figure 2 (a). Total cost is 28 units.

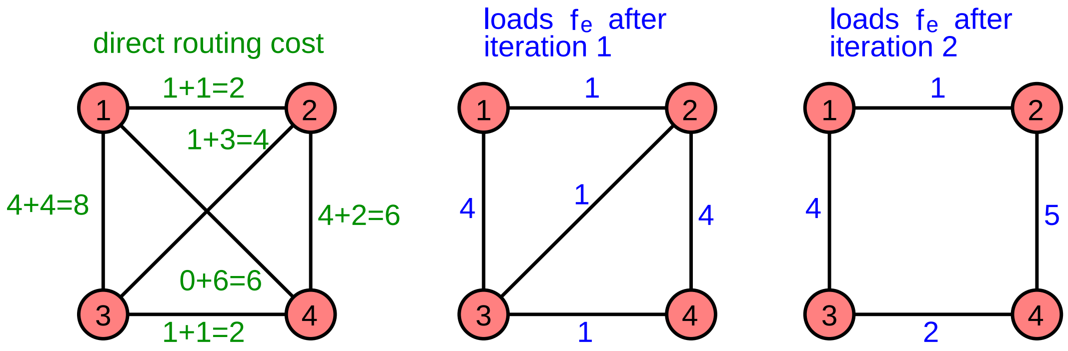

Figure 1: Minoux's greedy algorithm.

Iteration 1: Calculate the change in cost if each route is removed:

$$
\begin{aligned}
\Delta_{12} &= (\alpha_{13} + \alpha_{32} - \alpha_{12}) f_{12} - \beta_{12} = (1 + 1 - 1) \times 1 - 1 = 0 \\
\Delta_{13} &= (\alpha_{14} + \alpha_{34} - \alpha_{13}) f_{13} - \beta_{13} = (1 + 1 - 1) \times 4 - 4 = 0 \\
\Delta_{14} &= (\alpha_{12} + \alpha_{42} - \alpha_{14}) f_{14} - \beta_{14} = (1 + 1 - 1) \times 0 - 6 = -6 \\
\Delta_{23} &= (\alpha_{24} + \alpha_{34} - \alpha_{23}) f_{23} - \beta_{23} = (1 + 1 - 1) \times 1 - 3 = -2 \\
\Delta_{24} &= (\alpha_{12} + \alpha_{14} - \alpha_{24}) f_{24} - \beta_{24} = (1 + 1 - 1) \times 4 - 2 = 2 \\
\Delta_{34} &= (\alpha_{23} + \alpha_{24} - \alpha_{34}) f_{34} - \beta_{34} = (1 + 1 - 1) \times 1 - 1 = 0
\end{aligned}
$$

The minimum occurs for link $(1, 4)$, with a value of $-6 < 0$, so we remove this link, and the new cost will be $28 - 6 = 22$. Reroute the traffic from link $(1, 4)$ along path $1 - 2 - 4$ (Note there are two alternative paths here, the other is $1 - 3 - 4$). The new network loads are shown in Figure 2 (b). Note they are still just the direct routing loads, because there was no direct path traffic on link $1 - 4$.

Iteration 2: Calculate the change in cost if each route is removed:

$$
\begin{aligned}
\Delta_{12} &= (\alpha_{13} + \alpha_{32} - \alpha_{12}) f_{12} - \beta_{12} = (1 + 1 - 1) \times 1 - 1 = 0 \\
\Delta_{13} &= (\alpha_{14} + \alpha_{34} - \alpha_{13}) f_{13} - \beta_{13} = (1 + 1 - 1) \times 4 - 4 = 0 \\
\Delta_{23} &= (\alpha_{24} + \alpha_{34} - \alpha_{23}) f_{23} - \beta_{23} = (1 + 1 - 1) \times 1 - 3 = -2 \\
\Delta_{24} &= (\alpha_{12} + \alpha_{14} - \alpha_{24}) f_{24} - \beta_{24} = (1 + 1 - 1) \times 4 - 2 = 2 \\
\Delta_{34} &= (\alpha_{23} + \alpha_{24} - \alpha_{34}) f_{34} - \beta_{34} = (1 + 1 - 1) \times 1 - 1 = 0
\end{aligned}
$$

The minimum occurs for link $(2, 3)$, with a value of $-2 < 0$, so we remove this link, and the new cost will be $22 - 2 = 20$. Reroute the traffic from link $(2, 3)$ along path $2 - 4 - 3$ (note that there are actually two possible paths we might choose, the alternative is $2 - 1 - 3$). The new network loads are shown in Figure 2 (c).

Iteration 3: Calculate the change in cost if each route is removed:

$$\begin{aligned}
\Delta_{12} &= (\alpha_{13} + \alpha_{34} + \alpha_{41} - \alpha_{12}) f_{12} - \beta_{12} = (1+1+1-1) \times 1 - 1 = 1 \\
\Delta_{13} &= (\alpha_{12} + \alpha_{24} + \alpha_{43} - \alpha_{13}) f_{13} - \beta_{13} = (1+1+1-1) \times 4 - 4 = 4 \\
\Delta_{24} &= (\alpha_{21} + \alpha_{13} + \alpha_{34} - \alpha_{24}) f_{24} - \beta_{24} = (1+1+1-1) \times 5 - 2 = 8 \\
\Delta_{34} &= (\alpha_{31} + \alpha_{12} + \alpha_{24} - \alpha_{34}) f_{34} - \beta_{34} = (1+1+1-1) \times 2 - 1 = 3
\end{aligned}$$

All are positive, so the final network is the network shown in Figure 2 (c), with cost 20. This shows a heuristic solution to find the minimum cost solutoin. Because it is a heuristic, we do not know if it is the true minimum without doing more work (e.g. Branch and Bound).

Note alterantive Δ_e given alternative route choice at iteration 2 is

$$\begin{aligned}
\Delta_{12} &= 3 \\
\Delta_{13} &= 6 \\
\Delta_{24} &= 6 \\
\Delta_{34} &= 1
\end{aligned}$$

2. Apply Minoux's greedy algorithm to the networkshown in Figure 3 costs, and traffic as shown in the figure. Show all working! Note that for your convenience the costs and traffic are also defined in the matrices below.

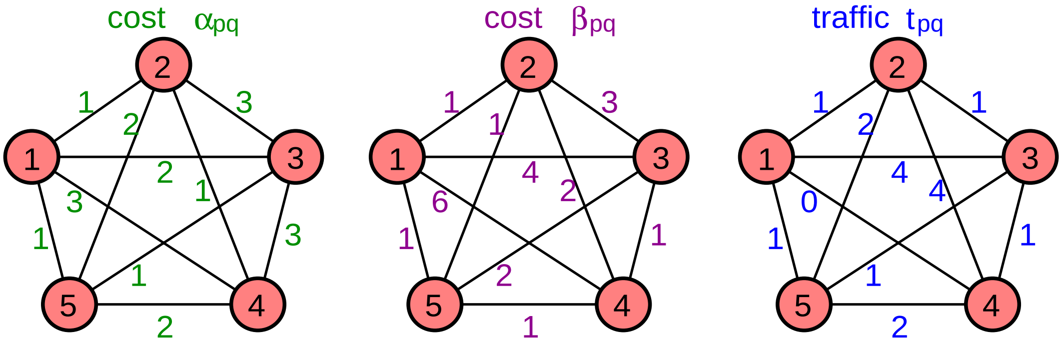

Figure 2: A network and associated costs and traffic.

$$\alpha = \begin{pmatrix} 0 & 1 & 2 & 3 & 1 \\ 1 & 0 & 3 & 1 & 2 \\ 2 & 3 & 0 & 3 & 1 \\ 3 & 1 & 3 & 0 & 2 \\ 1 & 2 & 1 & 2 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 & 4 & 6 & 1 \\ 1 & 0 & 3 & 2 & 1 \\ 4 & 3 & 0 & 1 & 2 \\ 6 & 2 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 4 & 0 & 1 \\ 1 & 0 & 1 & 4 & 2 \\ 4 & 1 & 0 & 1 & 1 \\ 0 & 4 & 1 & 0 & 2 \\ 1 & 2 & 1 & 2 & 0 \end{pmatrix}.$$

Iteration 1, current cost = 51.0

$$\begin{aligned}
\Delta_{1,2} &= 1.00 \\
\Delta_{1,3} &= -4.00 \\
\Delta_{1,4} &= -6.00 \\
\Delta_{1,5} &= 1.00 \\
\Delta_{2,3} &= -3.00 \\
\Delta_{2,4} &= 10.00 \\
\Delta_{2,5} &= -1.00 \\
\Delta_{3,4} &= -1.00
\end{aligned}$$

$$\begin{aligned}\Delta_{3,5} &= 0.00\\ \Delta_{4,5} &= 1.00\end{aligned}$$

The minimum $\Delta = -6.0$, for link (1,4)
The corresponding route is 4-2-1
The resulting links loads f are

$$f = \begin{pmatrix} 0.0 & 1.0 & 4.0 & 0.0 & 1.0 \\ 1.0 & 0.0 & 1.0 & 4.0 & 2.0 \\ 4.0 & 1.0 & 0.0 & 1.0 & 1.0 \\ 0.0 & 4.0 & 1.0 & 0.0 & 2.0 \\ 1.0 & 2.0 & 1.0 & 2.0 & 0.0 \end{pmatrix}$$

Iteration 2, current cost = 45.0

$$\begin{aligned}\Delta_{1,2} &= 1.00\\ \Delta_{1,3} &= -4.00\\ \Delta_{1,5} &= 1.00\\ \Delta_{2,3} &= -3.00\\ \Delta_{2,4} &= 10.00\\ \Delta_{2,5} &= -1.00\\ \Delta_{3,4} &= -1.00\\ \Delta_{3,5} &= 0.00\\ \Delta_{4,5} &= 1.00\end{aligned}$$

The minimum $\Delta = -4.0$, for link (1,3)
The corresponding route is 3-5-1
The resulting links loads f are

$$f = \begin{pmatrix} 0.0 & 1.0 & 0.0 & 0.0 & 5.0 \\ 1.0 & 0.0 & 1.0 & 4.0 & 2.0 \\ 0.0 & 1.0 & 0.0 & 1.0 & 5.0 \\ 0.0 & 4.0 & 1.0 & 0.0 & 2.0 \\ 5.0 & 2.0 & 5.0 & 2.0 & 0.0 \end{pmatrix}$$

Iteration 3, current cost = 41.0

$$\begin{aligned}\Delta_{1,2} &= 1.00\\ \Delta_{1,5} &= 9.00\\ \Delta_{2,3} &= -3.00\\ \Delta_{2,4} &= 10.00\\ \Delta_{2,5} &= -1.00\\ \Delta_{3,4} &= -1.00\\ \Delta_{3,5} &= 18.00\\ \Delta_{4,5} &= 1.00\end{aligned}$$

The minimum $\Delta = -3.0$, for link (2,3)
The corresponding route is 3-5-2

The resulting links loads f are

$$f = \begin{pmatrix} 0.0 & 1.0 & 0.0 & 0.0 & 5.0 \\ 1.0 & 0.0 & 0.0 & 4.0 & 3.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 6.0 \\ 0.0 & 4.0 & 1.0 & 0.0 & 2.0 \\ 5.0 & 3.0 & 6.0 & 2.0 & 0.0 \end{pmatrix}$$

Iteration 4, current cost = 38.0

$$\begin{aligned} \Delta_{1,2} &= 1.00 \\ \Delta_{1,5} &= 9.00 \\ \Delta_{2,4} &= 10.00 \\ \Delta_{2,5} &= -1.00 \\ \Delta_{3,4} &= -1.00 \\ \Delta_{3,5} &= 22.00 \\ \Delta_{4,5} &= 1.00 \end{aligned}$$

The minimum $\Delta = -1.0$, for link (3,4)
The corresponding route is 4-5-3
The resulting links loads f are

$$f = \begin{pmatrix} 0.0 & 1.0 & 0.0 & 0.0 & 5.0 \\ 1.0 & 0.0 & 0.0 & 4.0 & 3.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 7.0 \\ 0.0 & 4.0 & 0.0 & 0.0 & 3.0 \\ 5.0 & 3.0 & 7.0 & 3.0 & 0.0 \end{pmatrix}$$

Iteration 5, current cost = 37.0

$$\begin{aligned} \Delta_{1,2} &= 1.00 \\ \Delta_{1,5} &= 9.00 \\ \Delta_{2,4} &= 10.00 \\ \Delta_{2,5} &= -1.00 \\ \Delta_{3,5} &= Inf \\ \Delta_{4,5} &= 2.00 \end{aligned}$$

The minimum $\Delta = -1.0$, for link (2,5)
The corresponding route is 5-1-2
The resulting links loads f are

$$f = \begin{pmatrix} 0.0 & 4.0 & 0.0 & 0.0 & 8.0 \\ 4.0 & 0.0 & 0.0 & 4.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 7.0 \\ 0.0 & 4.0 & 0.0 & 0.0 & 3.0 \\ 8.0 & 0.0 & 7.0 & 3.0 & 0.0 \end{pmatrix}$$

Iteration 6, current cost = 36.0

$$\Delta_{1,2} = 11.00$$

$$
\begin{aligned}
\Delta_{1,5} &= 23.00 \\
\Delta_{2,4} &= 10.00 \\
\Delta_{3,5} &= Inf \\
\Delta_{4,5} &= 2.00
\end{aligned}
$$

The minimum $\Delta = 0.0 \geq 0$ so STOP

Final Solution: Cost = 36

$$
connectivity = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}
$$

$$
f = \begin{pmatrix} 0 & 4 & 0 & 0 & 8 \\ 4 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 7 \\ 0 & 4 & 0 & 0 & 3 \\ 8 & 0 & 7 & 3 & 0 \end{pmatrix}
$$

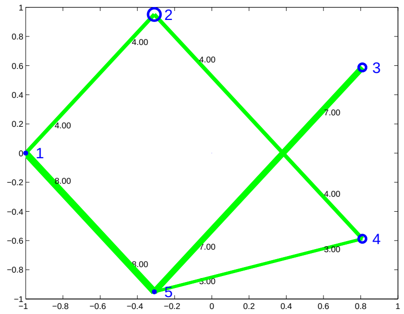

Figure 3: Final network for part 2.

In both cases the optimal solution is found, but this is not guaranteed by the algorithm.