Communications Network Design

lecture 12

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Budget constraint model and branch and bound

Branch and bound is a standard technique for solving integer programs, by relaxing the problem to the non-integer problem to find bounds, and using these to prune a tree of the possible solutions (rather than evaluating all possible solutions).

Budget Constraint Model

- separable linear cost model

$$
\begin{aligned}
C(\mathbf{f}) &= \sum_{e \in L(\mathbf{f})} (\beta_e + \alpha_e f_e) \qquad \text{where } L(\mathbf{f}) = \{e \in E : f_e > 0\} \\
&= \sum_{e \in L(\mathbf{f})} \beta_e + \sum_{\mu \in P} l_\mu(L(\mathbf{f}))\, x_\mu
\end{aligned}
$$

- separate costs into
  - initial **investment** costs (of laying optical fibre)

$$C_{\text{inv}}(L) = \sum_{e \in L} \beta_e$$

  - **operations** cost of lighting up the link

$$C_{\text{op}}(\mathbf{f}, L) = \sum_{e \in L} \alpha_e f_e$$

Budget Constraint Model (BCM)

- ealier, we considered the problem

$$\min C(\mathbf{f}) = \min \left[C_{\text{inv}}(L) + C_{\text{op}}(\mathbf{f}, L)\right]$$

  subject to the appropriate constraints

- budget constraint model
  - have a budget constraint on the investment costs

$$C_{\text{inv}}(L) \leq B$$

  - consider the optimization problem

$$\min C_{\text{op}}(\mathbf{f}, L) \text{ subject to } C_{\text{inv}}(L) \leq B$$

    with additional constraints as above.

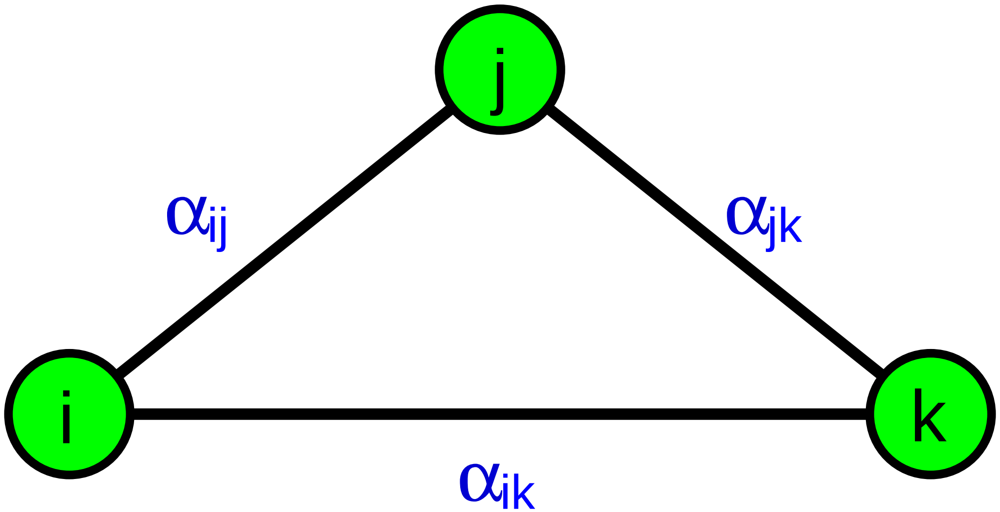
Formulation: of BCM

$$
\begin{array}{llrll}
\text{(P')} & \min & C(\mathbf{f}) & = \sum_{e \in L} \alpha_e f_e & \\
& \text{s.t.} & f_e & = \sum_{\mu : e \in \mu} x_\mu & \forall e \in E \\
& & \sum_{\mu : \mu \in P_k} x_\mu & = t_k & \forall k \in K \\
& & \sum_{e \in E} \beta_e z_e & \leq B & \\
& & x_\mu & \geq 0 & \forall \mu \in P \\
& & z_e & = 0, \text{ or } 1 & \forall e \in E
\end{array}
$$

$$
z_e = \begin{cases} 1 & \text{if link } e \in L \text{ (i.e. we use } e) \\ 0 & \text{if link } e \notin L \text{ (i.e. we don't use } e) \end{cases}
$$

BCM and the triangle inequality

- α_e satisfy the triangle inequality



$$\alpha_{ij} < \alpha_{ik} + \alpha_{kj}$$

- because β_e have been moved into constraints
- otherwise, link $e = (i, j)$ could be deleted as it is a longer path than $i - k - j$

BCM and Branch and Bound

- this is an old, well studied problem, e.g. see [1]
- NP-hard
- look for heuristic solutions
  - branch and bound [2]
- **Branch and Bound** is the topic of this lecture.

Notation

We can write an optimization problem several different ways

- integer linear programming problem, called (IP)

$$
\text{(IP)} \begin{cases} \text{maximize} & \mathbf{c}^T\mathbf{x} & & \\ \text{subject to} & A\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & 0 \\ & \mathbf{x} & \in & \mathbb{Z}^n \end{cases}
$$

- short form

$$
\max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0, \mathbf{x} \in \mathbb{Z}^n\}
$$

Integer programming

- Take an integer linear programming problem

$$\max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0, \mathbf{x} \in \mathbb{Z}^n\}$$

  - some of our variables are real (e.g. link loads)
    - we have a **mixed**-integer linear programming problem
  - $\mathbb{Z}^n$ is the set of n-dimensional vectors of integers
    - we will restrict to $\mathbf{x} \in \{0,1\}^n$
  - Many other classic examples
    - travelling salesman problem
    - knapsack problem
    - set covering problem
    - machine scheduling problem

Converting BCM into integer program

Variables are

$$z_e = \begin{cases} 1 & \text{if link } e \in L \text{ (i.e. we use } e\text{)} \\ 0 & \text{if link } e \notin L \text{ (i.e. we don't use } e\text{)} \end{cases}$$

Write optimization objective

$$\begin{aligned} C(\mathbf{f}) &= \sum_e \alpha_e f_e && (1) \\ &= \sum_e \alpha_e \sum_{\mu: e \in \mu} x_\mu && (2) \\ &= \sum_e \sum_\mu \alpha_e A(e,\mu) x_\mu && (3) \\ &= [\alpha^t A]\,\mathbf{x} && (4) \end{aligned}$$

Converting BCM into integer program

We derive the routing vector $\mathbf{x}$ from the $\mathbf{z}$ by solving the shortest path problem (with linear costs) on the graph determined by the $\mathbf{z}$.

Converting BCM into integer program

Obvious constraints given in the BCM are

$$\sum_{\mu:\mu\in P_k} x_\mu = t_k, \quad \forall k \in K \tag{5}$$

$$\sum_{e\in E} \beta_e z_e \leq B \tag{6}$$

we just need to write these in matrix form, but there is a less obvious contraint

$$(1 - z_e) f_e = (1 - z_e) \sum_{\mu:e\in\mu} x_\mu = 0 \tag{7}$$

which says we cannot put traffic on absent links.

Relationship to linear programming

For each integer program:

(IP) $\max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0, \mathbf{x} \in \mathbb{Z}^n\}$

there is an associated linear program:

(LP) $\max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0\}$

Now (LP) is less constrained than (IP) so

- If (LP) is infeasible, then so is (IP)
- If (LP) is optimized by integer variables, then that solution is feasible and optimal for (IP)
- The optimal objective value for (LP) is greater than or equal to the optimal objective for (IP)

Bounds

- call the (LP) a **relaxation**
  - because we have relaxed some constraints
- it is easy to solve (usually)
  - its a standard linear program
  - can use simplex, or interior point methods
- rounding off the solution to the relaxation might work badly
  - it could even produce a partitioned graph
  - not all traffic can get through!
- but the (LP) relaxation does provide a **bound**
  - we can use this to **prune branches**

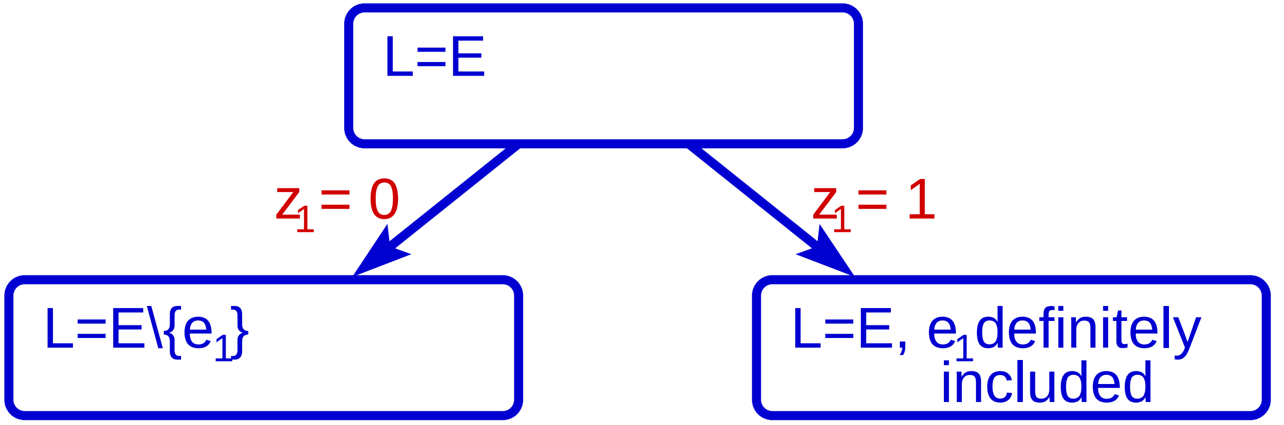
Branching

- the above gives us bounds for solutions
- we also need to **branch**
  - at each point where we don't have an integer solution, we can branch by splitting the possible solutions into two partitions
  - for example, we require $x_1 \in \{0, 1\}$, but the relaxation solution was $x_1 = 0.2$, we then subdivide the problem into two parts
    - $x_1 = 0$
    - $x_1 = 1$
  - then solve each of these subproblems

Branching example

For the network problem, we have decision variables

$$z_e = \begin{cases} 1 & \text{if link } e \in L \text{ (i.e. we use } e\text{)} \\ 0 & \text{if link } e \notin L \text{ (i.e. we don't use } e\text{)} \end{cases}$$



Branching example

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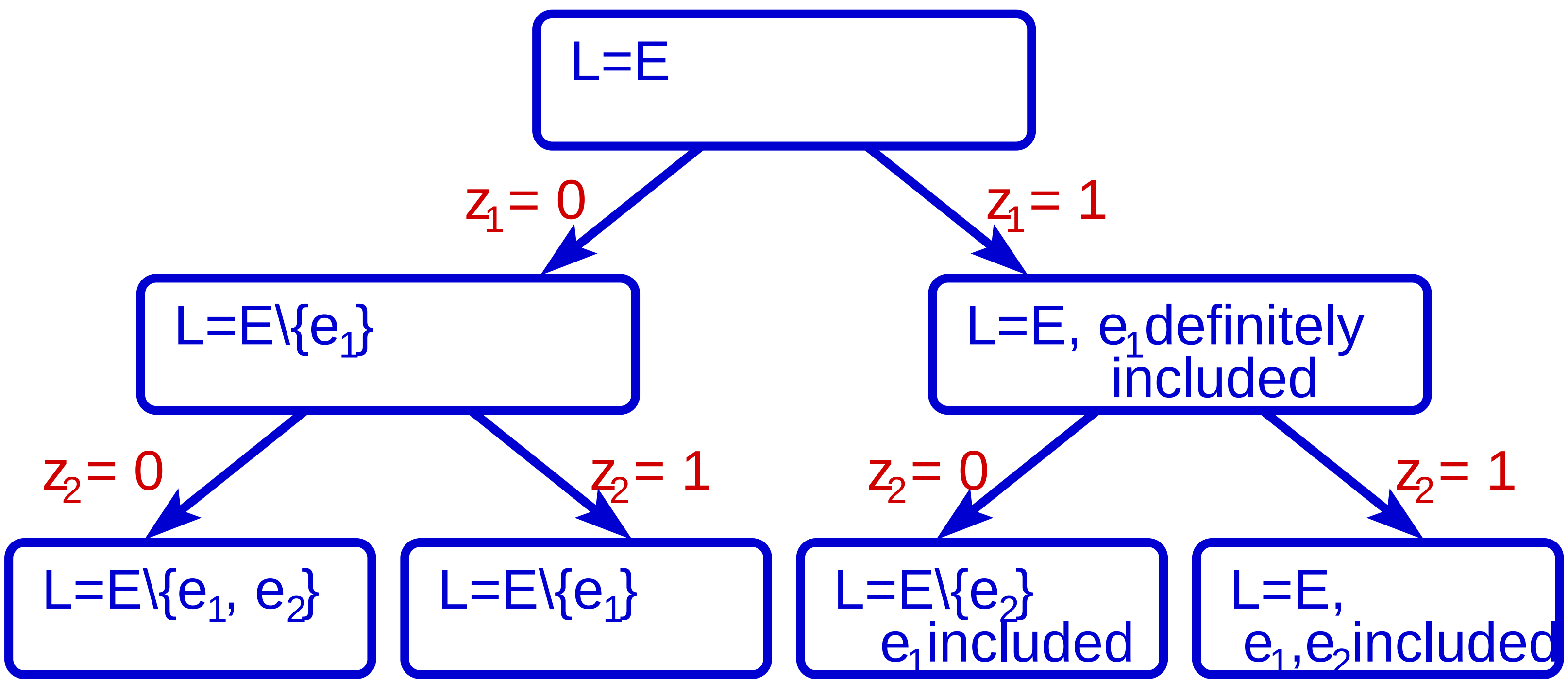

Branch and Bound

- **key:** if upper bound of a subproblem is less than objective for a known integer feasible solution, then
  - the subproblem cannot have a solution greater than the already known solution
  - we can eliminate this solution
  - we can also prune all of the tree below the solution
- it lets us do a **non-exhaustive** search of the subproblems
  - if we get to the end, we have a proof of optimality without exhaustive search

Branch and Bound: algorithm

1. **Initialization:** initialize variables, in particular, start a list of subproblems, initialized with our original integer program.
2. **Termination:** terminate the program when we reach the optimum (i.e. the list of subproblems is empty).
3. **Problem selection and relaxation:** select the next problem from the list of possible subproblems, and solve a relaxation on it.
4. **Fathoming and pruning:** eliminate branches of the tree once we prove they cannot contain an optimal solution.
5. **Branching:** partition the current problem into subproblems, and add these to our list.

Branch and Bound: example

Consider the problem (from [2])

$$
IP^0 \left\{ \begin{array}{lrcl} \text{maximize} & 13x_1 + 8x_2 & & \\ \text{subject to} & x_1 + 2x_2 & \leq & 10 \\ & 5x_1 + 2x_2 & \leq & 20 \\ & x_1 \geq 0, x_2 \geq 0 & & \\ & x_1, x_2 \text{ integer} & & \end{array} \right.
$$

Branch and Bound: algorithm

Initialization:

- initialize the **list** of problems $\mathcal{L}$
  - set initially $\mathcal{L} = \{\text{IP}^0\}$, where IP^0 is the initial problem
  - often store/picture $\mathcal{L}$ as a tree
- incumbent objective value $z_{ip} = -\infty$
- initial value of upper bound on problem is $\overline{z}_0 = \infty$
- constraint set of problem IP^0 is set to be $S^0 = \{\mathbf{x} \in \mathbb{Z}^n | A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0\}$

Branch and Bound: algorithm

Termination:

- If $\mathcal{L} = \phi$ then we stop
  - If $z_{ip} = -\infty$ then the integer program is infeasible.
  - Otherwise, the subproblem IP^i which yielded the current value of z_{ip} is optimal gives the optimal solution $\mathbf{x}^*$

Branch and Bound: algorithm

Problem selection:

- select a problem from $\mathcal{L}$
  - there are multiple ways to decide which problem to choose from the list
    - the method used can have a big impact on speed
  - once selected, delete the problem from the list

Relaxation:

- solve a relaxation of the problem
  - denote the optimal solution by $\mathbf{x}^{iR}$
  - denote the optimal objective value by z_i^R
    - $z_i^R = -\infty$ if no feasible solutions exist

Branch and Bound: algorithm

Fathoming :

- we say branch of the tree is **fathomed** if
  - infeasible
  - feasible solution, and $z_i^R \leq z_{ip}$
  - integral feasible solution
    - set $z_{ip} \leftarrow \max\{z_{ip}, z_i^R\}$

Pruning:

- in any of the cases above, we need not investigate any more subproblems of the current problem
  - subproblems have more constraints
  - their z must lie under the upper bound
- Prune any subtrees with $z_j^R \leq z_{ip}$
- If we pruned **Goto step 2**

Branch and Bound: algorithm

Branching:

- also called partitioning
- want to partition the current problem into subproblems
  - there are several ways to perform partitioning
- If S^i is the current constraint set, then we need a disjoint partition $\{S^{ij}\}_{j=1}^{k}$ of this set
  - we add problems $\{\text{IP}^{ij}\}_{j=1}^{k}$ to $\mathcal{L}$
  - IP^{ij} is just IP^{i} with its feasible region restricted to S^{ij}
- **Goto step 2**

Branch and Bound: algorithm

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Branch and Bound: example

Consider the problem (from [2])

$$\text{IP}^0 \begin{cases} \text{maximize} & 13x_1 + 8x_2 \\ \text{subject to} & x_1 + 2x_2 \leq 10 \\ & 5x_1 + 2x_2 \leq 20 \\ & x_1 \geq 0, x_2 \geq 0 \\ & x_1, x_2 \text{ integer} \end{cases}$$

with relaxation

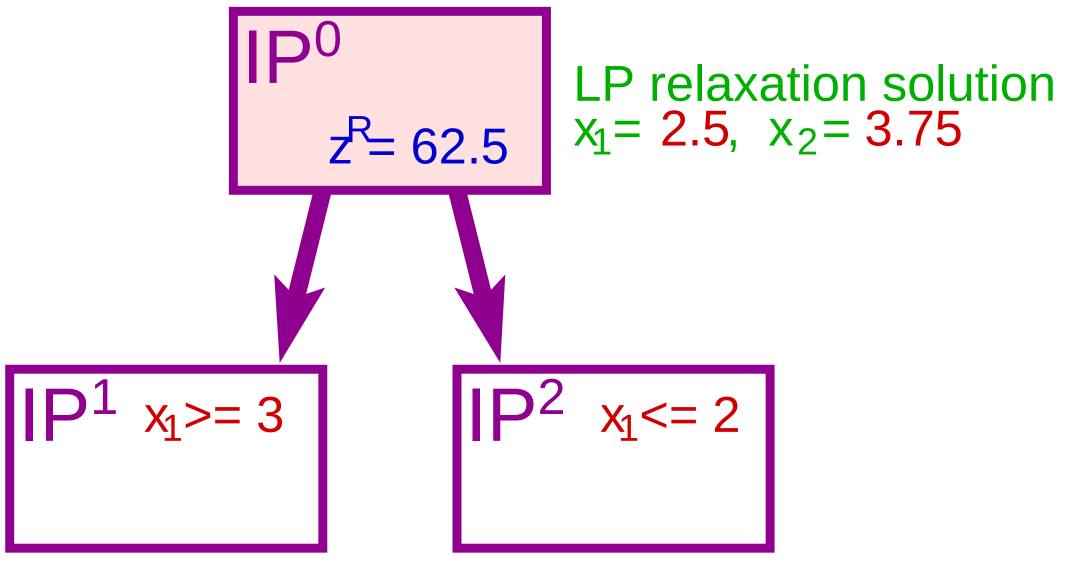
$$\text{LP}^0 \begin{cases} \text{maximize} & z = 13x_1 + 8x_2 \\ \text{subject to} & x_1 + 2x_2 \leq 10 \\ & 5x_1 + 2x_2 \leq 20 \\ & x_1 \geq 0, x_2 \geq 0 \end{cases}$$

which has solutions $x_1^0 = 2.5$ and $x_2^0 = 3.75$ with $z_0^R = 62.5$

Branch and Bound: example

- we will partition on x_1
  - this is the "most infeasible"
    - furthest from an integral value
- we will partition on x_1
  - partition into two subproblems by adding an extra constraint
    - IP^1 has $x_1 \geq 3$
    - IP^2 has $x_1 \leq 2$
- $\mathcal{L} = \{\text{IP}^1, \text{IP}^2\}$

Branch and Bound: example



$\mathcal{L} = \{\mathtt{IP}^1, \mathtt{IP}^2\}$

Branch and Bound: example

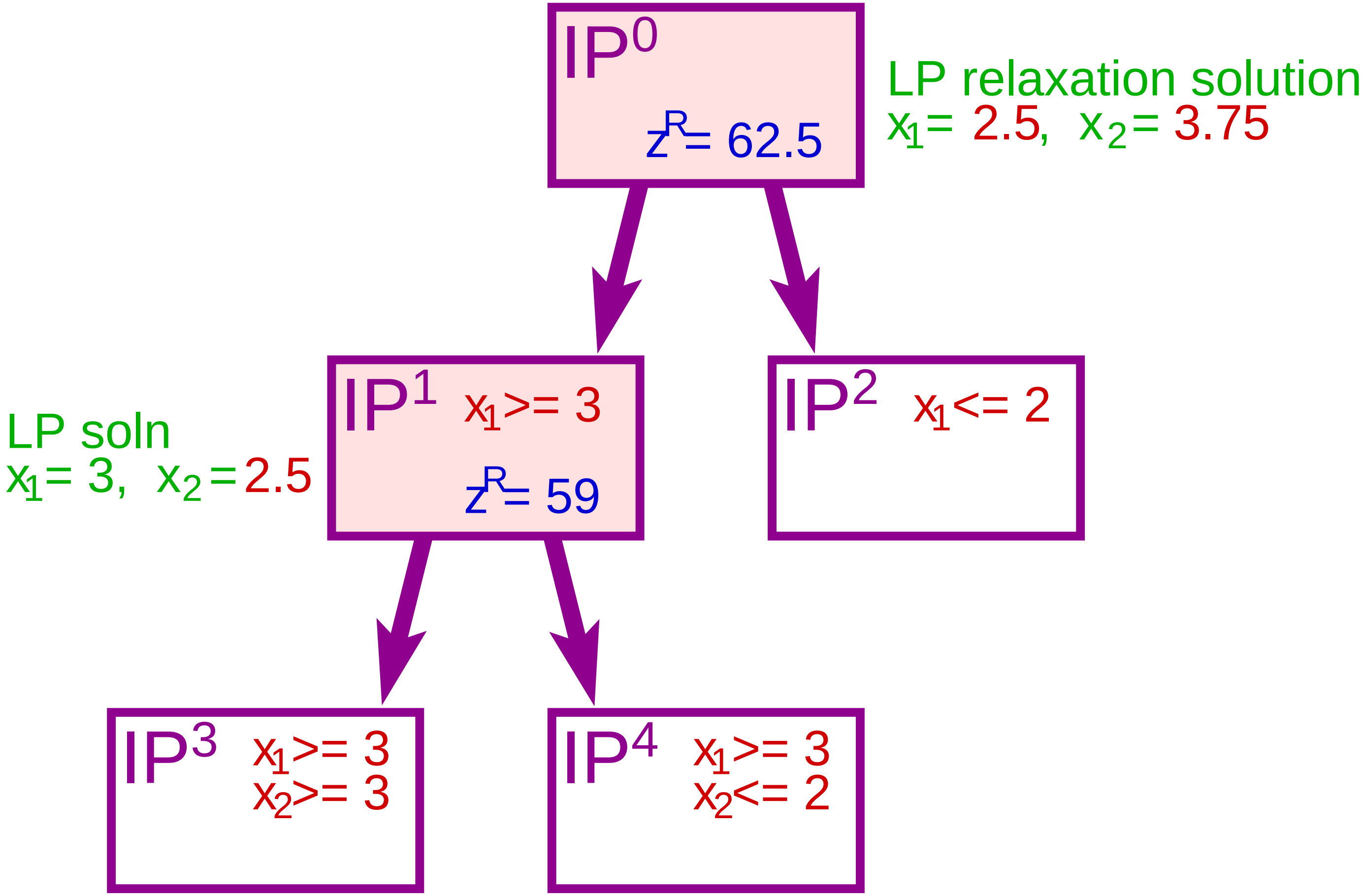
Problem selection (just chose in order) of IP^1

$$IP^1 \begin{cases} \text{maximize} & 13x_1 + 8x_2 \\ \text{subject to} & x_1 + 2x_2 \leq 10 \\ & 5x_1 + 2x_2 \leq 20 \\ & x_1 \geq 3 \\ & x_1 \geq 0, x_2 \geq 0 \\ & x_1, x_2 \text{ integer} \end{cases}$$

The relaxation (to a LP) has solutions

- $x_1^1 = 3$ and $x_2^1 = 2.5$ with $z_1^R = 59$
- we will next partition on x_2
  - IP^3 has $x_2 \leq 2$
  - IP^4 has $x_2 \geq 3$

Branch and Bound: example



$$\mathcal{L} = \{\mathtt{IP}^2, \mathtt{IP}^3, \mathtt{IP}^4\}$$

Branch and Bound: example

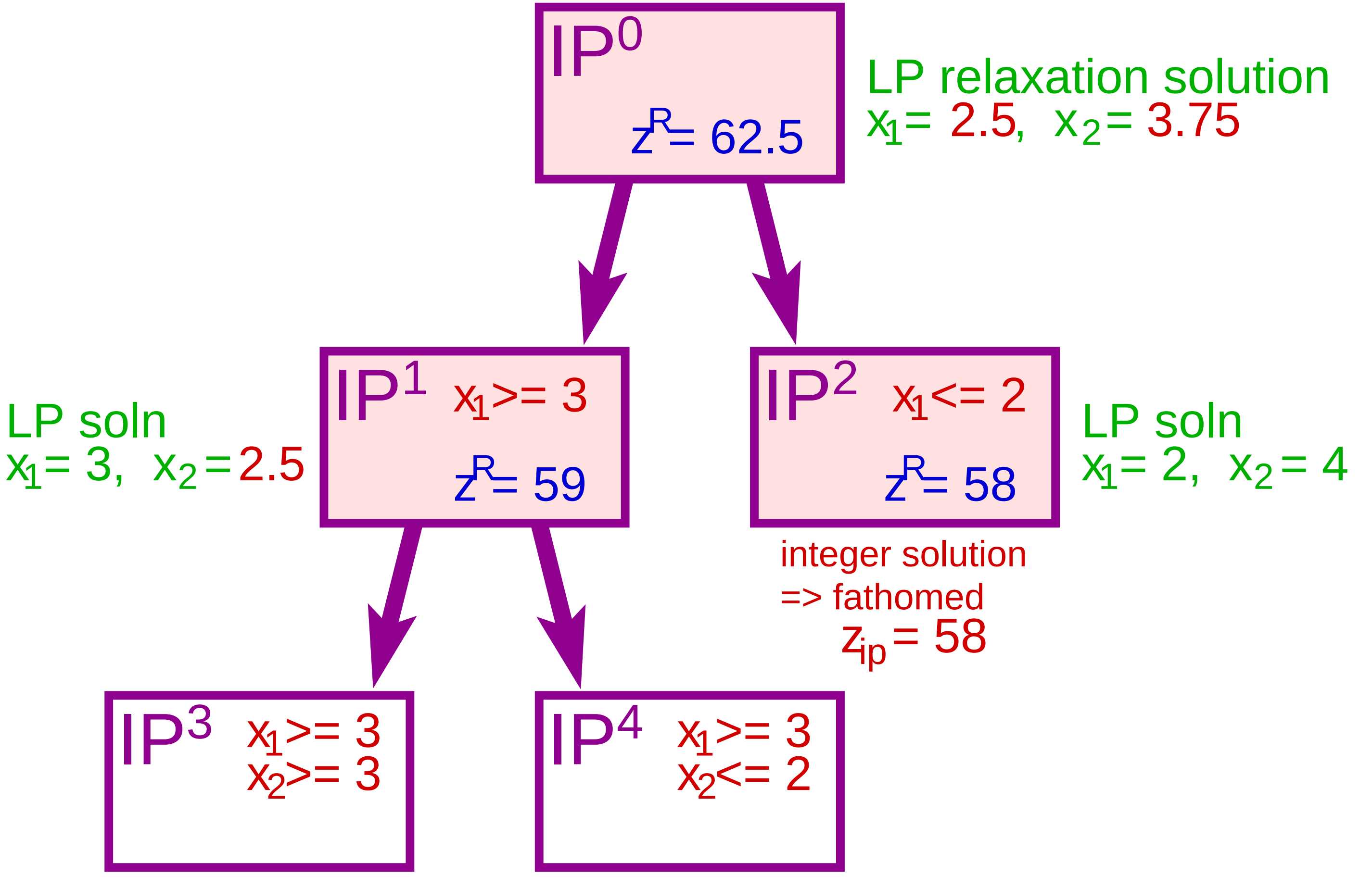
Problem selection (best bound) of IP^2

$$IP^2 \left\{ \begin{array}{lrcl} \text{maximize} & 13x_1 + 8x_2 & & \\ \text{subject to} & x_1 + 2x_2 & \leq & 10 \\ & 5x_1 + 2x_2 & \leq & 20 \\ & x_1 & \leq & 2 \\ & x_1 \geq 0, x_2 \geq 0 & & \\ & x_1, x_2 \text{ integer} & & \end{array} \right.$$

The relaxation (to a LP) has solutions

- $x_1^2 = 2$ and $x_2^2 = 4$ with $z_2^R = 58$
- integral feasible
- $z_{ip} = 58$
- IP^2 is fathomed

Branch and Bound: example



$\mathcal{L} = \{\mathtt{IP}^3, \mathtt{IP}^4\}$

Branch and Bound: example

Problem selection (order) of IP^3

$$\text{IP}^3 \begin{cases} \text{maximize} & 13x_1 + 8x_2 & & \\ \text{subject to} & x_1 + 2x_2 & \leq & 10 \\ & 5x_1 + 2x_2 & \leq & 20 \\ & x_1 & \geq & 3 \\ & x_2 & \geq & 3 \\ & x_1 \geq 0, x_2 \geq 0 & & \\ & x_1, x_2 \text{ integer} & & \end{cases}$$

The relaxation (to a LP) is infeasible

- $z_3^R = -\infty$
- IP^3 is fathomed
- $\mathcal{L} = \{\text{IP}^4\}$

Branch and Bound: example

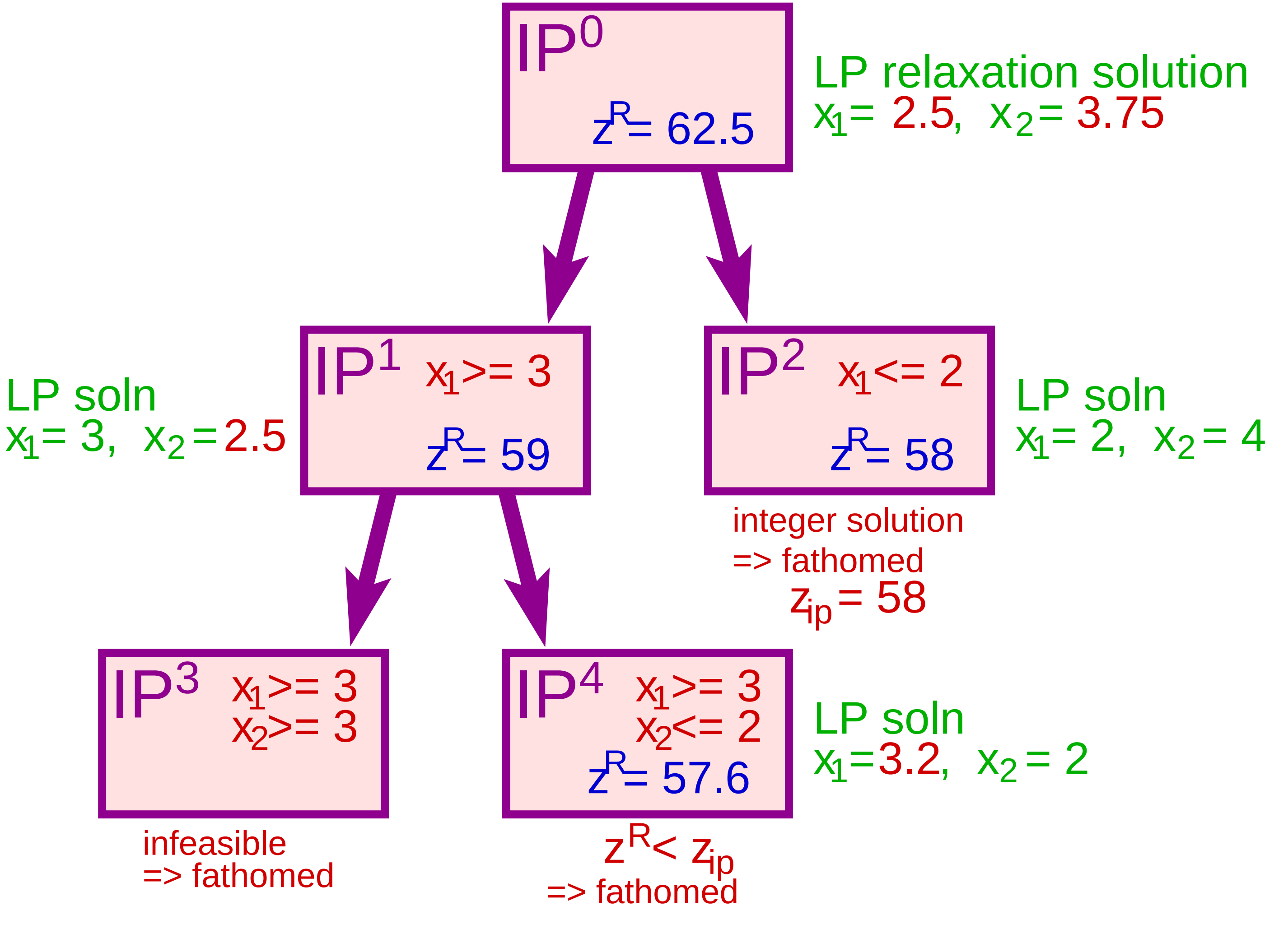
Problem selection (only possible one) of IP^4

$$IP^4 \begin{cases} \text{maximize} & 13x_1 + 8x_2 \\ \text{subject to} & x_1 + 2x_2 \leq 10 \\ & 5x_1 + 2x_2 \leq 20 \\ & x_1 \geq 3 \\ & x_2 \leq 2 \\ & x_1 \geq 0, x_2 \geq 0 \\ & x_1, x_2 \text{ integer} \end{cases}$$

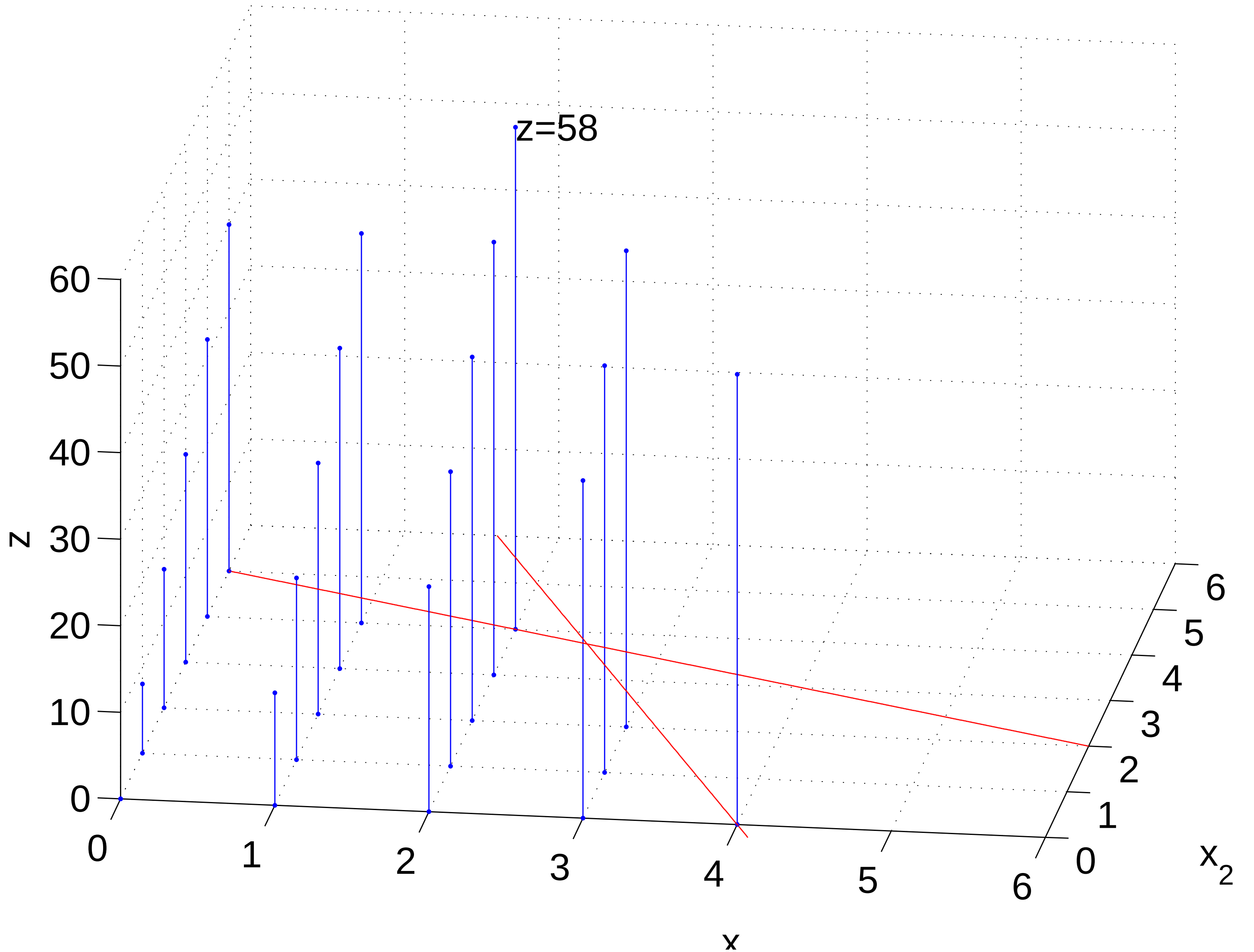
The relaxation (to a LP) has solution

- $x_1^2 = 3.2$ and $x_2^2 = 2$ with $z_4^R = 57.6 < z_{ip}$
- IP^4 is fathomed

Branch and Bound: example



Branch and Bound: example



Branch and Bound

- B&B is a very general algorithm
  - as described above we seek the optimum
  - can also be used as a heuristic
- different strategies available for each step above
  - can use heuristics inside B&B
  - pre-processing of the problem can be good
- no single strategy stands out as best for all problems
  - but sometimes we can exploit properties of a particular problem to do better

References

[1] D. S. Johnson, J. K. Lenstra, and A. H. G. R. Kan, "The complexity of the network design problem," Networks, vol. 8, pp. 279–285, 1978.

[2] E. K. Lee and J. Mitchell, Encyclopedia of Optimization, ch. Branch-and-bound methods for integer programming. Kluwer Academic Publishers, 2001. http://www.rpi.edu/~mitchj/papers/leeejem.html.