Information Theory and Networks

Lecture 5: Entropy

Matthew Roughan
<matthew.roughan@adelaide.edu.au>
http://www.maths.adelaide.edu.au/matthew.roughan/Lecture_notes/InformationTheory/

School of Mathematical Sciences,
University of Adelaide

September 18, 2013

Part I

Entropy

> You should call it entropy, for two reasons. In the first place your uncertainty function has been used in statistical mechanics under that name, so it already has a name. In the second place, and more important, no one really knows what entropy really is, so in a debate you will always have the advantage.
>
> *John von Neumann, Suggesting to Claude Shannon a name for his new uncertainty function, as quoted in Scientific American Vol. 225 No. 3, (1971), p. 180*

Section 1

Entropy: definitions

Entropy

Entropy will be our measure of uncertainty.

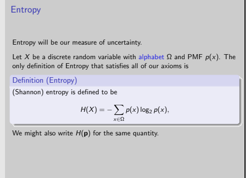
Let X be a discrete random variable with alphabet Ω and PMF $p(x)$. The only definition of Entropy that satisfies all of our axioms is

Definition (Entropy)

(Shannon) entropy is defined to be

$$H(X) = -\sum_{x\in\Omega} p(x)\log_2 p(x),$$

We might also write $H(\mathbf{p})$ for the same quantity.



The usual convention is that $p\log p = 0$ when $p = 0$ for the purpose of this definition, justified by taking limits as $p \to 0$.

It is conventional to take logs base 2, but any other base would work, just the units would differ.

Note, that as we already stated

$$H(X) = -E\left[\log_2(p(X))\right] = E\left[\log_2\left(\frac{1}{p(X)}\right)\right].$$

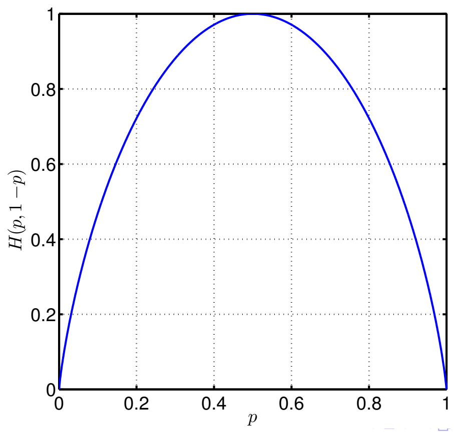
Entropy Example 1: Bernoulli RV

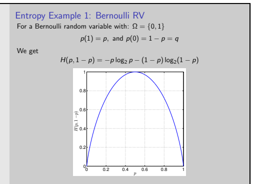
For a Bernoulli random variable with: $\Omega = \{0,1\}$

$$p(1) = p, \text{ and } p(0) = 1-p = q$$

We get

$$H(p, 1-p) = -p\log_2 p - (1-p)\log_2(1-p)$$





Entropy Example 2: [CT91, p.14]

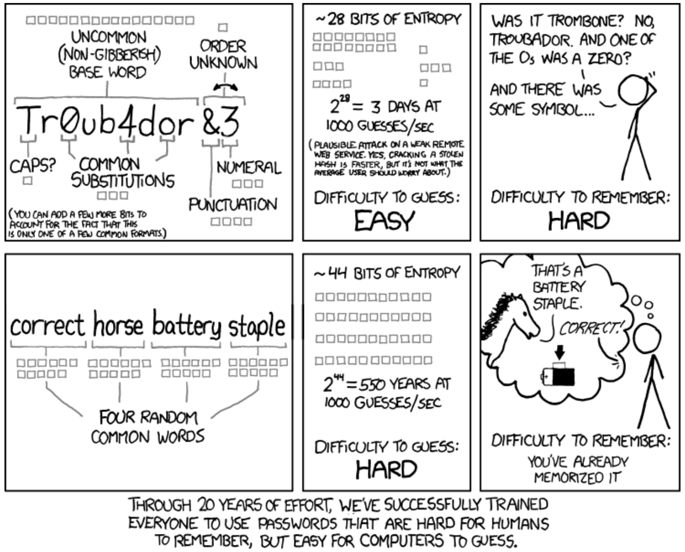
Take symbols $\Omega = \{a, b, c, d\}$, with probabilities

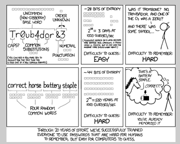
$$X = \begin{cases} a, & \text{with probability } 1/2, \\ b, & \text{with probability } 1/4, \\ c, & \text{with probability } 1/8, \\ d, & \text{with probability } 1/8, \end{cases}$$

Then

$$H(X) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{4}\log_2\frac{1}{4} - \frac{1}{8}\log_2\frac{1}{8} - \frac{1}{8}\log_2\frac{1}{8} = \frac{7}{4} \text{ bits.}$$

Entropy Example 3: `https://xkcd.com/936/`





Joint Entropy

Definition (Joint Entropy)

Given two discrete RVs X and Y with joint distribution $p(x, y)$ the joint entropy is defined to be

$$H(X, Y) = -\sum_x \sum_y p(x, y) \log_2 p(x, y),$$

This shouldn't be surprising, as its just the same definition on the alphabet (x, y).

Joint Entropy
Definition (Joint Entropy)
Given two discrete RVs X and Y with joint distribution $p(x, y)$ the joint entropy is defined to be
$H(X, Y) = -\sum_x \sum_y p(x, y) \log_2 p(x, y).$
This shouldn't be surprising, as its just the same definition on the alphabet (x, y).

Conditional Entropy

Definition (Conditional Entropy)

Given two discrete RVs X and Y with joint distribution $p(x, y)$ the conditional entropy of Y given X is defined to be

$$\begin{aligned} H(Y|X) &= -E\left[\log p(Y|X)\right] \\ &= -\sum_x \sum_y p(x, y) \log p(y|x) \\ &= -\sum_x p(x) \sum_y p(y|x) \log p(y|x) \\ &= -\sum_x p(x) H(Y|X = x). \end{aligned}$$

where $p(x)$ is the marginal distribution of X.

Conditional Entropy
Definition (Conditional Entropy)
Given two discrete RVs X and Y with joint distribution $p(x, y)$ the conditional entropy of Y given X is defined to be
$H(Y|X) = -E[\log p(Y|X)] = -\sum_x \sum_y p(x, y) \log p(y|x) = -\sum_x p(x) \sum_y p(y|x) \log p(y|x) = -\sum_x p(x) H(Y|X = x).$
where $p(x)$ is the marginal distribution of X.

Conditional Entropy: examples

- Perfect dependence: $Y = f(X)$, so given X there is no uncertainty about Y, then
$$H(Y|X) = 0$$
- Independence: $p(y|x) = p(y)$, so
$$H(Y|X) = H(Y),$$
so uncertainty of Y is unchanged by knowledge of X.

Relative Entropy

Relative entropy is an asymmetric measure of

- the “distance” between two distributions
- inefficiency of assuming q when p is true

Definition (Relative entropy)

The relative entropy or Kullback-Leibler divergence is a measure of the distance from PMF $p(x)$ to PMF $q(x)$ and is defined by
$$D(p\|q) = \sum_x p(x) \log \frac{p(x)}{q(x)} = E_p\left[\log \frac{p(X)}{q(X)}\right].$$

Again use the convention that $0 \log 0 = 0$, but also we will take $p \log \frac{p}{0} = \infty$.

Note that we specify that the expectation in the RHS of the definition is with respect to the probability distribution p. So this is not a true distance metric (in the mathematical sense) because it is not symmetric.

Later we will see how this is useful, when we look at mutual information and the Kraft-McMillan theorem.

Relative Entropy Example: [CT91, p.17]

Let (X, Y) have the following values

		X			
		1	2	3	4
	1	1/8	1/16	1/32	1/32
Y	2	1/16	1/8	1/32	1/32
	3	1/16	1/16	1/16	1/16
	4	1/4	0	0	0

The marginal distributions are

$$\begin{aligned} p(X) &= (1/2, 1/4, 1/8, 1/8) \\ p(Y) &= (1/4, 1/4, 1/4, 1/4) \end{aligned}$$

so $H(X) = 7/4$ and $H(Y) = 2$ bits.

Relative Entropy Example: [CT91, p.17]
Let (X, Y) have the following values
The marginal distributions are
so $H(X) = 7/4$ and $H(Y) = 2$ bits.

More Complex Entropy Example: [CT91, p.17]

Joint entropy

$$H(X, Y) = -\sum_x \sum_y p(x, y) \log_2 p(x, y) = 27/8 \text{ bits}$$

Conditional entropies

$$H(Y|X) = -\sum_x p(x) H(Y|X = x) = 13/8 \text{ bits}$$

$$H(X|Y) = -\sum_y p(y) H(X|Y = y) = 11/8 \text{ bits}$$

More Complex Entropy Example: [CT91, p.17]
Joint entropy
Conditional entropies

```
% function entropy_ex
%      Example, 2.2.1, from Cover and Thomas, p.17
clear;

PXY = [[1/8, 1/16, 1/32, 1/32];
       [1/16, 1/8, 1/32, 1/32];
       [1/16, 1/16, 1/16, 1/16];
       [1/4, 0, 0, 0]
      ]
PX = sum(PXY)          % marginal distributions
PY = sum(PXY')         % marginal distributions
1
HX = entropy(PX)       % marginal entropy
HY = entropy(PY)       % marginal entropy
HXY = entropy(PXY)     % joint entropy

% conditional entropy
HXgY = ( PY(1) * entropy(PXY(1,:)/sum(PXY(1,:))) ...
 + PY(2) * entropy(PXY(2,:)/sum(PXY(2,:))) ...
 + PY(3) * entropy(PXY(3,:)/sum(PXY(3,:))) ...
 + PY(4) * entropy(PXY(4,:)/sum(PXY(4,:))) )
HYgX = ( PX(1) * entropy(PXY(:,1)/sum(PXY(:,1))) ...
 + PX(2) * entropy(PXY(:,2)/sum(PXY(:,2))) ...
 + PX(3) * entropy(PXY(:,3)/sum(PXY(:,3))) ...
 + PX(4) * entropy(PXY(:,4)/sum(PXY(:,4))) )
```

Section 2

A Brief History of Information Theory

Section 2
A Brief History of Information Theory

Entropy

- First though about in the context of physics: statistical mechanics
  - Ludwig Boltzmann, 1872
  - J. Willard Gibbs, 1878
- More on that connection later

Entropy
- First though about in the context of physics: statistical mechanics
  - Ludwig Boltzmann, 1872
  - J. Willard Gibbs, 1878
- More on that connection later

Context

Telephone and Telegraphy grows massively

- Invented 1753?
- Concrete idea Samuel Soemmering in 1809
- Morse and Vail (not just code) 1835
- First serious demonstrator: Washington to Baltimore, a distance of 40 miles, was completed in 1844
  - The first message, composed by Annie Ellsworth, the young daughter of Morse's friend was "What hath God wrought?".
- First undersea cable Sept 1851 across English channel
- 1865 there were 83,000 miles of wire in the USA.
- First transatlantic line 1866
- Society of Telegraph Engineers was founded in 1871
- Todd's telegraphs importance to Australia 1872
- 1882 Bell Lab is created
- 1904 photograph transmitted by wire in Germany
- 1907, the US alone had around 3 million miles of telephone and telegraph wires
- The figure was 67.8 million miles by 1925

http://en.wikipedia.org/wiki/Electrical_telegraph
http://www.nadcomm.com/timeline.htm

Perhaps the peak of telegraphy was the 40s, particularly during the war, but after this the telephone started to take over.

Information

- 1924, Harry Nyquist starts formalising transmission capacities
  - "intelligence" and the speed it can be transmitted
- 1928, Ralph Hartley introduces Hartley Information, as log of number of possible messages (or log of alphabet size)
- 1940, Alan Turing introduces the deciban in relationship to finding cypher settings

Hartley:

- Intelligence in the sense of "military intelligence" or information
- An axiom we didn't think about was that information shouldn't really depend on the alphabet, so if you changed your symbols, then that shouldn't change the entropy. But Hartley's measure depended on the number of symbols, even if one isn't used.

Shannon and Information Theory

- By the 1940, AT&T/Bell had
  - nearly 100 million miles of telephone and telegraph cable
  - 280,000 employees
  - 80 million daily telephone calls
  - $1.2 billion revenue
- Along comes Shannon (joins Bell Labs in 1942)
  - worked for AT&T/Bell
  - influenced by
    - at princeton: Hermann Weyl, von Neumann, Einstein, Gödel
    - doing crypto during war: Alan Turing
    - MIT: Vannevar Bush
  - thought about TV, genes, cryptography, ...
- Newton made force into a quantity with units, Shannon made information into a quantity with units (bits)

Shannon and Information Theory
By the 1940, AT&T/Bell had
nearly 100 million miles of telephone and telegraph cable
280,000 employees
80 million daily telephone calls
$1.2 billion revenue
Along comes Shannon (joins Bell Labs in 1942)
worked for AT&T/Bell
influenced by
at princeton: Hermann Weyl, von Neumann, Einstein, Gödel
doing crypto during war: Alan Turing
MIT: Vannevar Bush
thought about TV, genes, cryptography, ...
Newton made force into a quantity with units, Shannon made information into a quantity with units (bits)

Later developments

- 1944, Shannon's theory mainly complete, but main publication in 1948
- 1947, Hamming codes
- 1949, Fano proves some basic results
- 1951, Relative entropy by Kullback and Leibler
- 1950s onwards various people used the ideas for coding (Huffman, Reed, Muller, Solomon, Gallager, Viterbi, ...)
- 1957, Jayne relates information theory back to statistics and physics

Later developments
1944, Shannon's theory mainly complete, but main publication in 1948
1947, Hamming codes
1949, Fano proves some basic results
1951, Relative entropy by Kullback and Leibler
1950s onwards various people used the ideas for coding (Huffman, Reed, Muller, Solomon, Gallager, Viterbi, ...)
1957, Jayne relates information theory back to statistics and physics

Today

More important than ever

- Mp3, video, voice, ...
- Internet
- Digital TV and radio
- Bio*informatics*
- Google and Big Data

Over 1.5 billion miles of "telephone" wire are said now to be strung across the U.S.

Anything you do as a scientist or mathematician will be influenced by information theory, whether you know it or not.

Today
More important than ever
- Mp3, video, voice, ...
- Internet
- Digital TV and radio
- Bioinformatics
- Google and Big Data
Over 1.5 billion miles of "telephone" wire are said now to be strung across the U.S.
Anything you do as a scientist or mathematician will be influenced by information theory, whether you know it or not.

See for more detail
http://en.wikipedia.org/wiki/Timeline_of_information_theory
http://en.wikipedia.org/wiki/History_of_entropy
http://www.historyofinformation.com/index.php?category=Communication+F+Information+Theory
[Gle11]

Further reading I

- Thomas M. Cover and Joy A. Thomas, *Elements of information theory*, John Wiley and Sons, 1991.
- James Gleick, *The information: a history, a theory, a flood*, Fourth Estate, 2011.