**Assignment 7**: Due Thursday 23rd May at 5pm

Late assignments will not be accepted except by prior arrangement (for a good reason)

**Please include your student number in your handed up work, as Canvas doesn't give this to me automatically.**

1. Given a semiring $(S, \oplus, \otimes, \bar{0}, \bar{1})$, define $M_n(S)$ to be the set of square $n \times n$ matrices, with elements from $S$, such that
   - $A \hat{\oplus} B$ is element-wise addition
   $$\left[A \hat{\oplus} B\right]_{ij} = a_{ij} \oplus b_{ij}$$
   - $A \hat{\otimes} B$ is the generalisation of standard matrix multiplication
   $$\left[A \hat{\otimes} B\right]_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes b_{kj}$$
   - and identities of the form, *e.g.*,
   $$\mathbf{0} = \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{bmatrix}$$

   Show that

   (a) $\hat{\otimes}$ is associative; and

   (b) idempotence of $\otimes$ does not imply idempotence of $\hat{\otimes}$.

   [5 marks]

2. Write (computer) functions implementing $\oplus$ and $\otimes$, *i.e.*, `oplus(A,B)` and `otimes(A,B)` for $A, B \in M_n(S)$, where $S$ is the Viterbi, or Max-times Semiring.

   (a) Use your functions to find $A^*$ (by fixed-point iteration or otherwise) the solution $Y$ to the equation
   $$Y = (A \otimes Y) \oplus \mathbf{I}$$
   for
   $$A = \begin{pmatrix} 0.0 & 0.3 & 0.9 & 0.1 \\ 0.5 & 0.0 & 0.5 & 0.5 \\ 0.9 & 0.2 & 0.0 & 0.1 \\ 0.1 & 0.6 & 0.0 & 0.0 \end{pmatrix}$$

   [3 marks]

   (b) Explain the meaning of this result.

   [1 mark]

   (c) What does it mean that $A^*$ is not symmetric?

   [1 mark]