Complex-Network Modelling and Inference

Lecture 16: Operations on graphs (unary operators)

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Operations on graphs

Operations of graphs are important for a number of reasons

- We can use them to build new graph models
- We can calculate properties of graphs
- We use them in proofs of graph properties

Think of them of constructing a grammar or an algebra of graphs.

Operators on graphs

- Types of operators
  1. operators that calculate properties of graphs (*e.g.*, metrics)
  2. operators that produce a new graph
  3. operators that work on weighted graphs to calculate new weights
- extra notation: for $G = (N, E)$, we define

$$\begin{aligned} N(G) &= N \\ E(G) &= E \end{aligned}$$

  i.e., $N(G)$ is the nodes of G and $E(G)$ the edges.
- need to start by defining isomorphic graphs

Graph Isomorphism (reminder)

- First need to know when graphs are the "same"
- Labels often don't matter (or aren't known)
- Two graphs G and H are *isomorphic* if there exists a bijection f between the nodes of G and H

$$f : N(G) \to N(H)$$

such that it preserves adjacency, *i.e.*,

$$(u, v) \in E(G) \Leftrightarrow (f(u), f(v)) \in E(H)$$

- call the bijection (function) f an isomorphism
- We write two graphs are isomorphic as $G \simeq H$

Section 1

Unary operators

Unary Operators

Operations that map G to G'

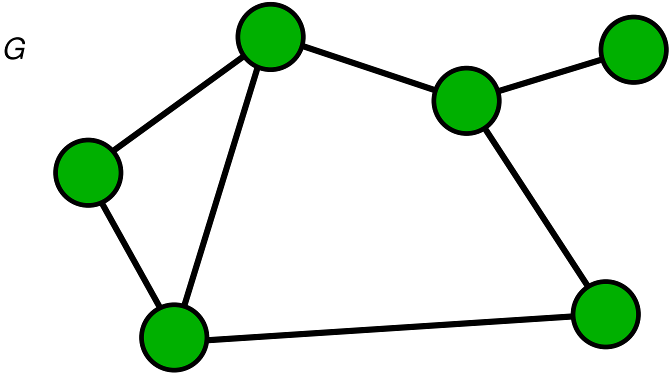
- Complement G^C
- Transpose G^T of a digraph
- Line graph $L(G)$ of graph G
- Power G^k, for $k = 1, 2, \ldots$
- Subdivision
- Others
  - Graph Minor
  - Mycielskian

Complement G^C

- $N(G^C) = N(G)$ and

$$e \in E(G^C) \Leftrightarrow e \notin E(G)$$

- *e.g.,*



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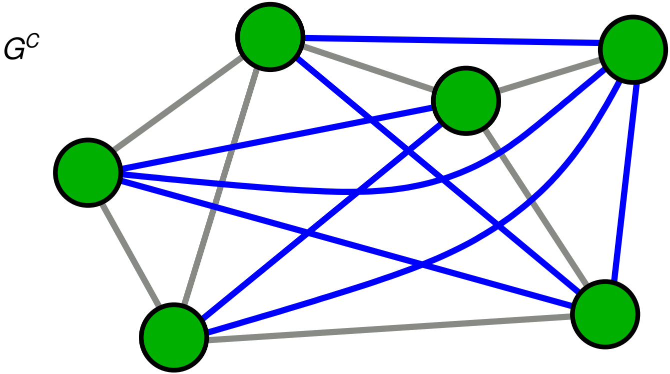
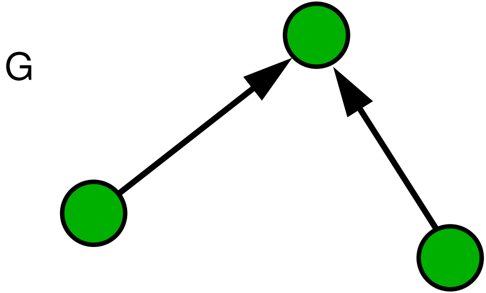


Transpose G^T

- Adjacency matrix is transposed
- Reverse directions of links (in digraph)
- Also called converse, or reverse



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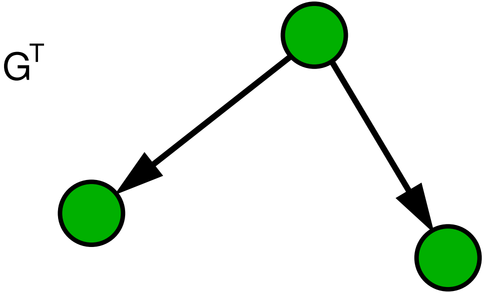


Line graph $L(G)$

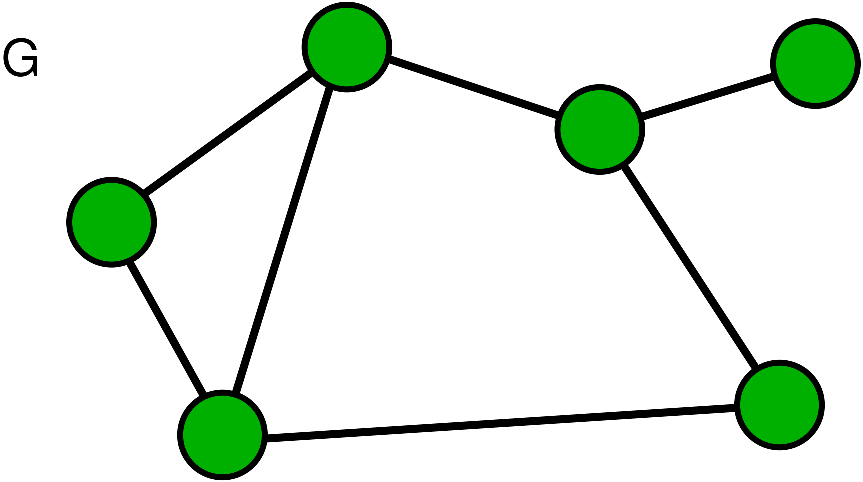
- Sometimes called adjoint, conjugate, edge-to-vertex dual, ...
- Every edge becomes a node
- Node in $L(G)$ a adjacent if the corresponding edges in G share a common end-point.
- Formally:

$$G = (N, E) \Rightarrow L(G) = \left(E, E'\right)$$

where

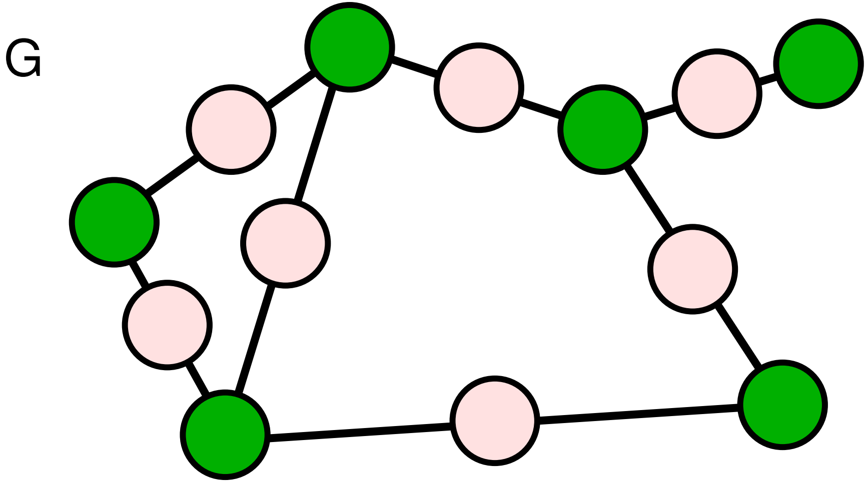
$$((i,j),(k,m)) \in E' \Leftrightarrow (i = k) \vee (i = m) \vee (j = k) \vee (j = m)$$

Example Line Graph



Each node in G creates a little clique in $L(G)$.

Example Line Graph



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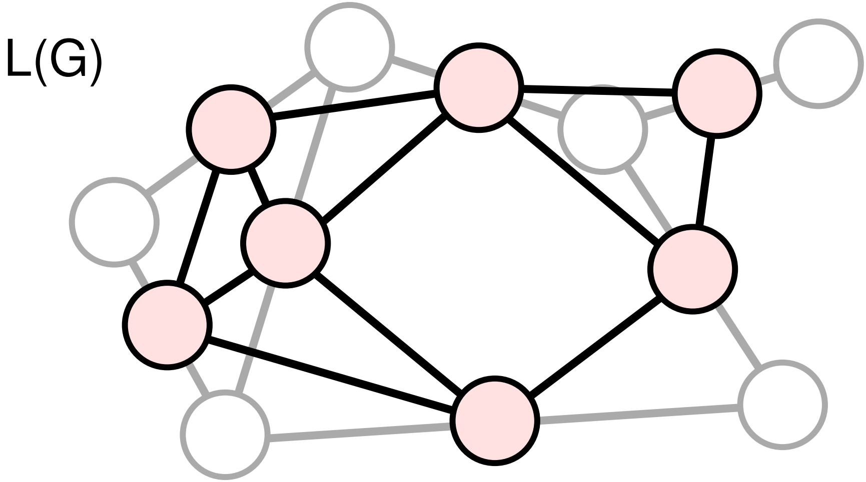

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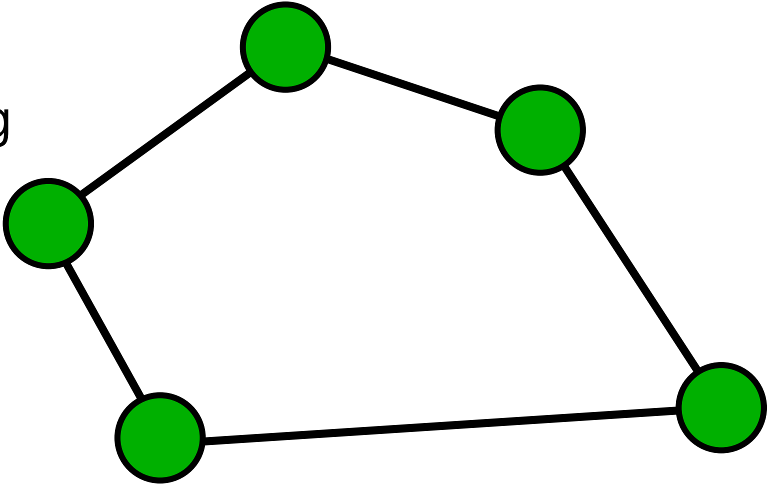
Properties of Line Graph

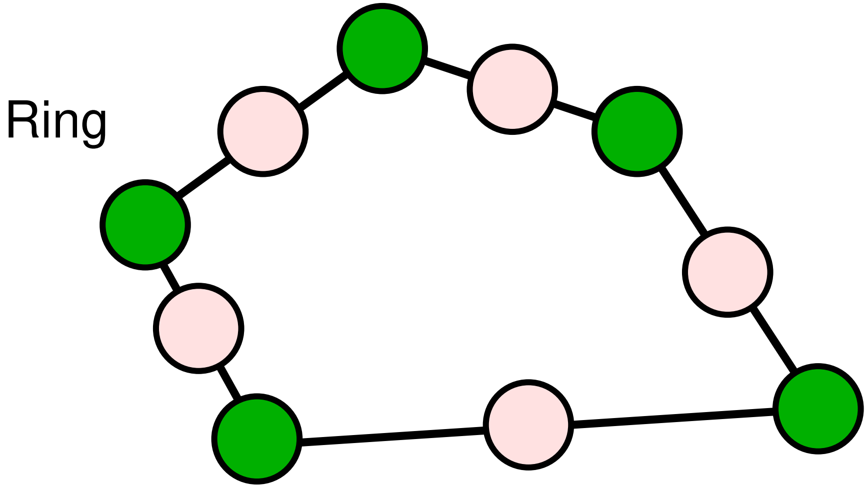
- If G is connected, then $L(G)$ is connected
  - converse is not true
- not all graphs are a line graph
- for a finite connected graph the sequence $G, L(G), L(L(G)), L(L(L(G))), \ldots$ has only 4 cases
  - If G is a cycle graph then they are all isomorphic
  - If G is a path graph then each subsequent graph is a shorter path until eventually the sequence terminates with an empty graph.
  - If G is a star with 4 nodes, then all subsequent graphs are triangles
  - The graphs in the sequence increase indefinitely

Line Graph: case 1

Ring



Line Graph: case 1



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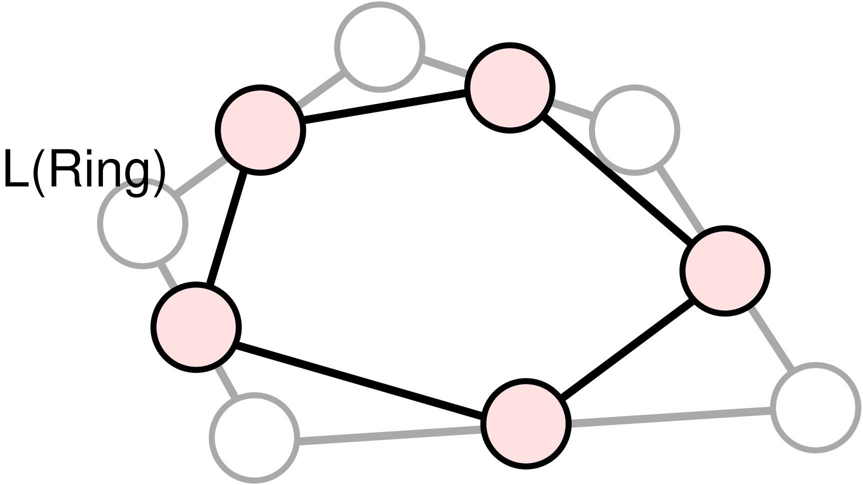
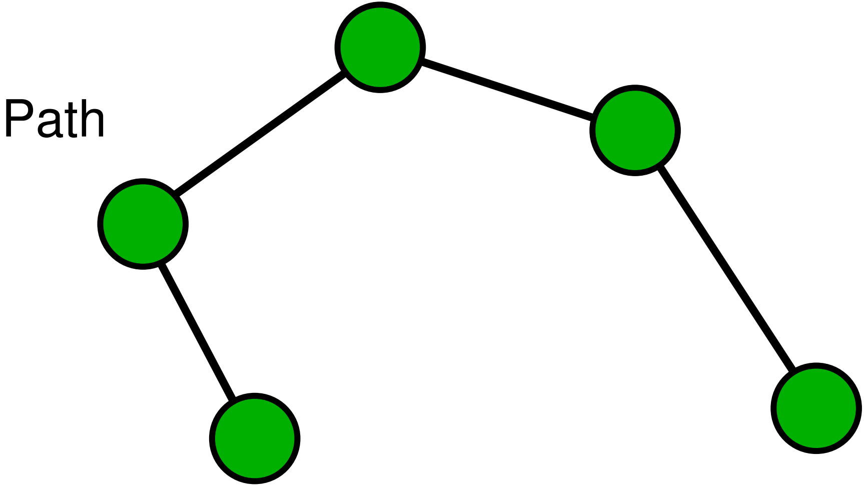
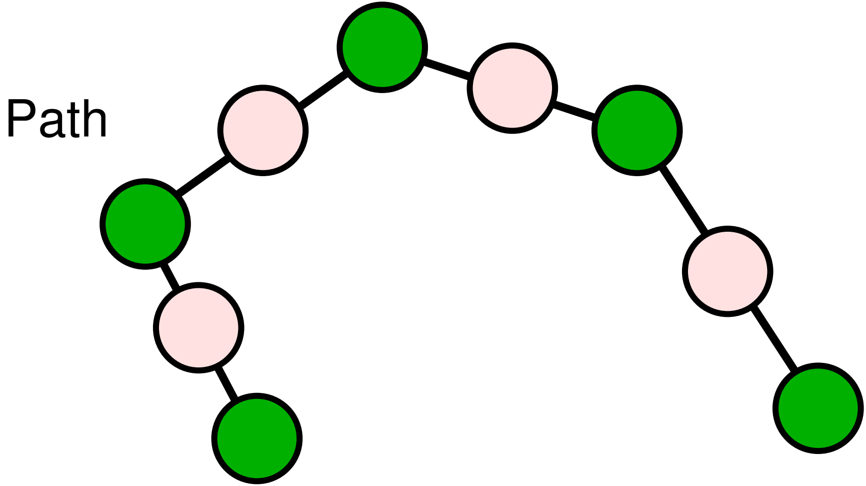


Line Graph: case 2



The line graph of a ring is an isomorphic ring (a ring with the same number of nodes).

Line Graph: case 2



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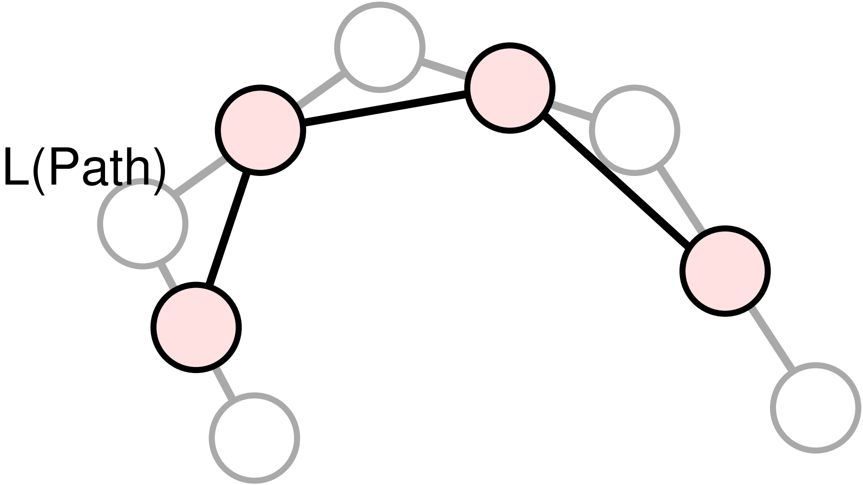
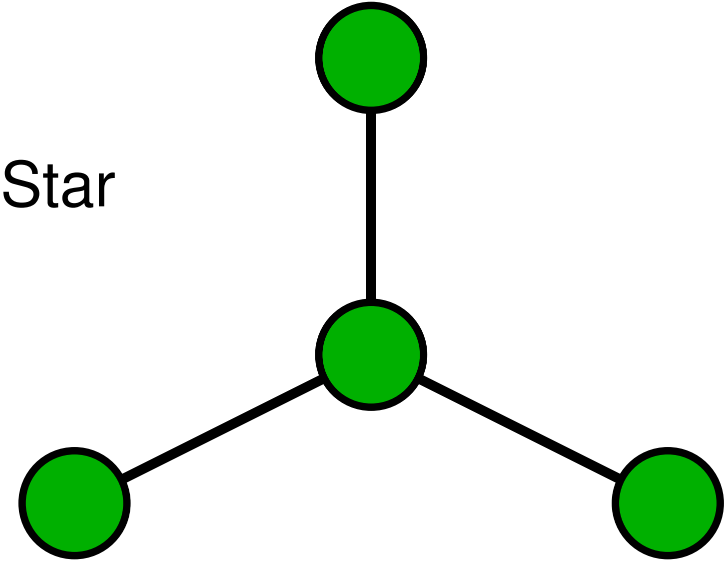


The line graph of a ring is an isomorphic ring (a ring with the same number of nodes).

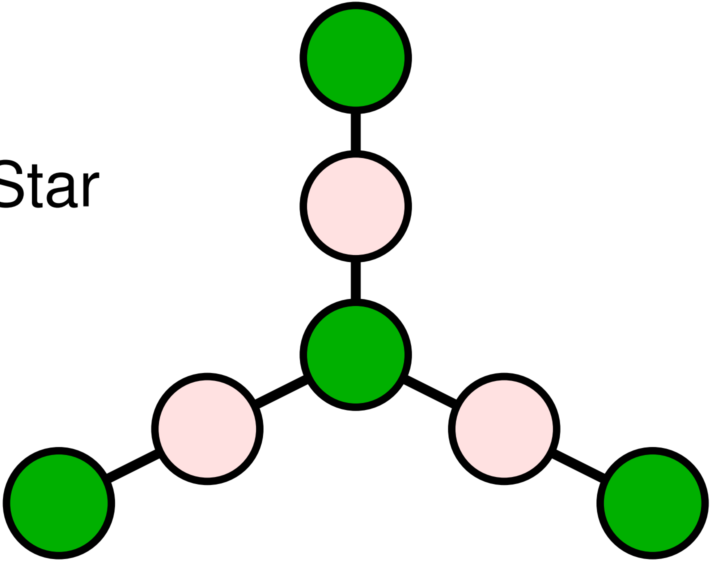
Line Graph: case 3



The line graph of a 4 node star is a 3 node ring (a triangle). Using case 1

Line Graph: case 3

Star



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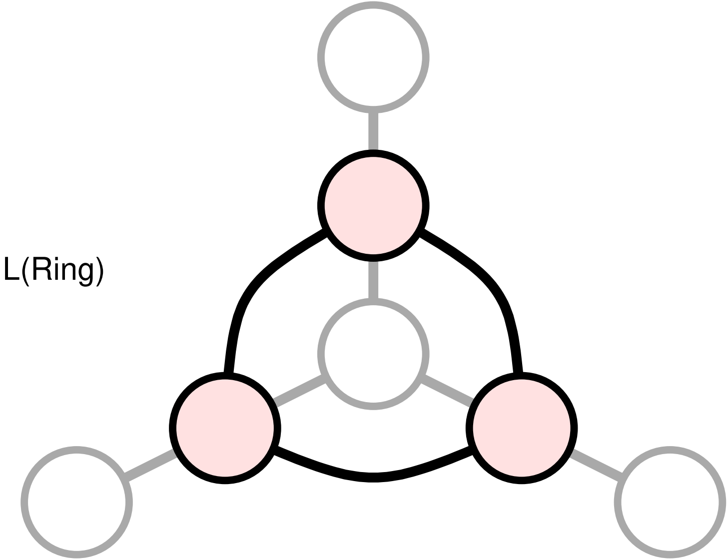

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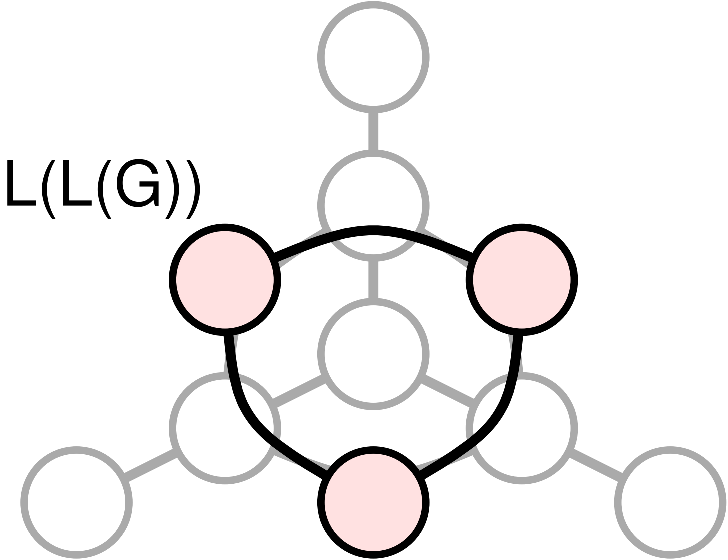

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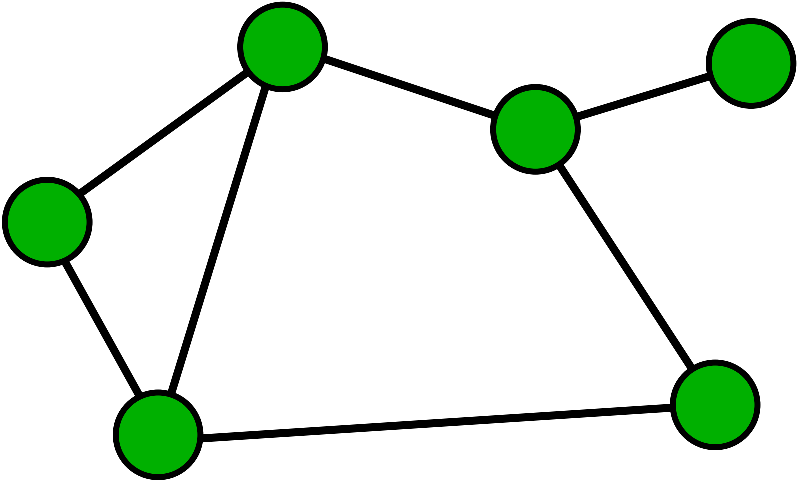
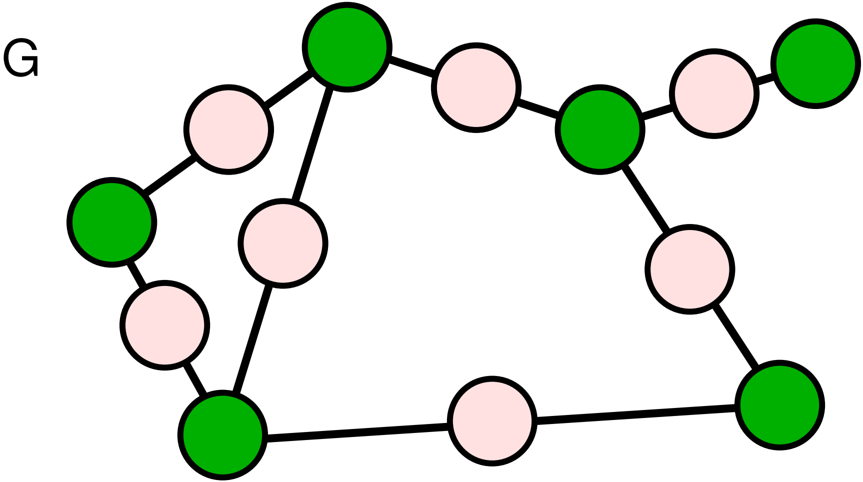
Line Graph: case 4

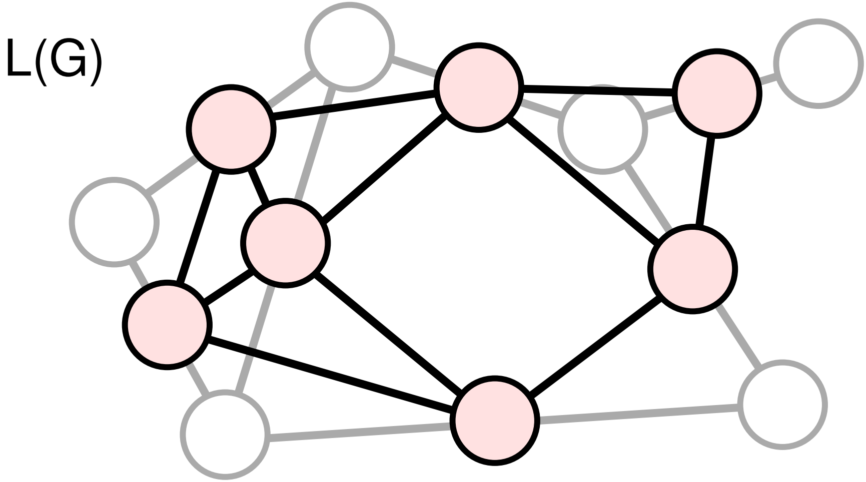
G



Line Graph: case 4



Line Graph: case 4



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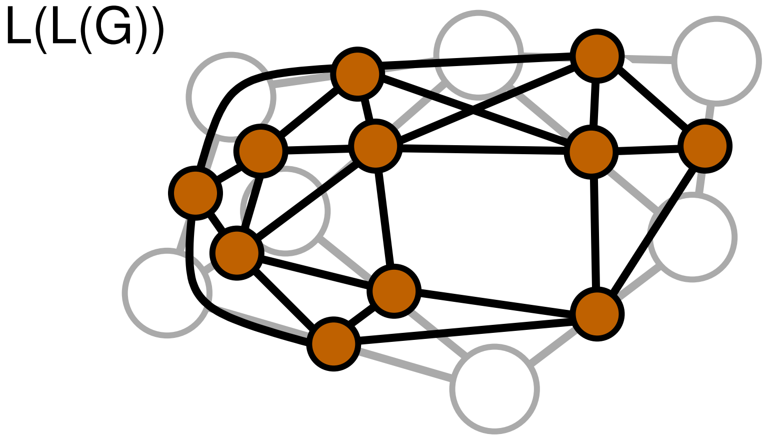


Line Graph growth

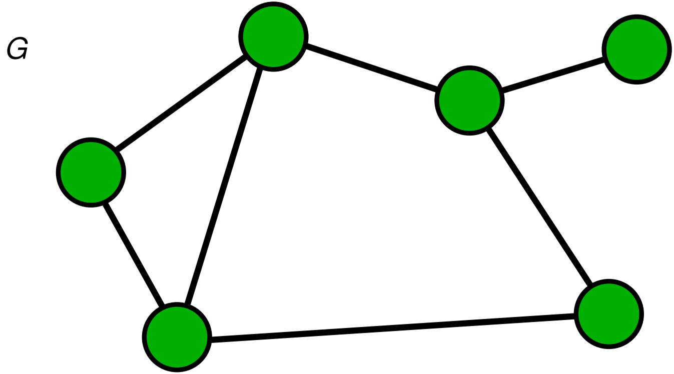
If G has n nodes, and e edges, then $L(G)$ has $n' = e$ nodes and e' edges where

$$e' = \frac{1}{2} \sum_{i=1}^{n} k_i^2 - e$$

where k_i are the node degrees

Graph Power G^k

- G^k is the graph formed from the nodes of G, and with edges between all pairs of nodes with (hop) distance no more than k.
- For example:



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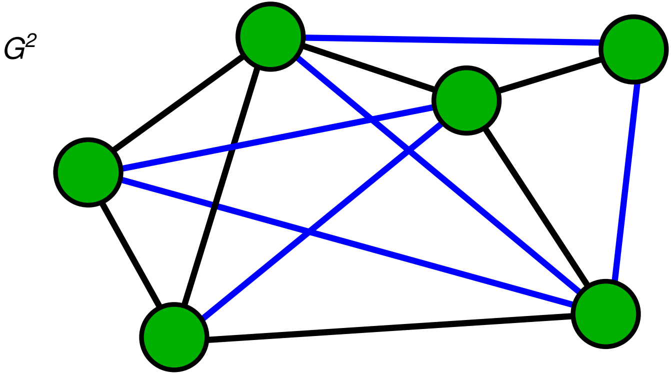

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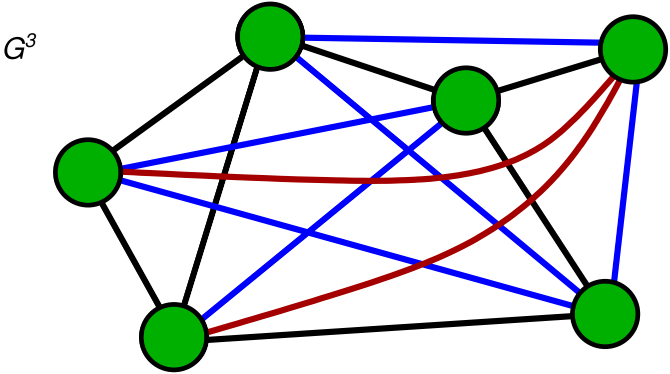


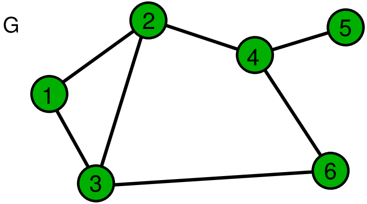
Graph-Power Adjacency Matrix

- We can obtain the adjacency matrix of a graph power G^k, by taking the sum of the first kth powers of the adjacency matrix of G, and thresholding,
- *i.e.*,

$$A^{(k)} = I\left[\left(\sum_{i=1}^{k} A^i\right) > 0\right]$$

  - A is the adjacency matrix of a graph power G
  - $A^{(k)}$ is the adjacency matrix of a graph power G^k
  - $I(\cdot)$ is an indicator function, applied elementwise to the matrix.
- NB: Element (i, j) in A^k counts the **number** of paths of length k between i and j in the original graph.

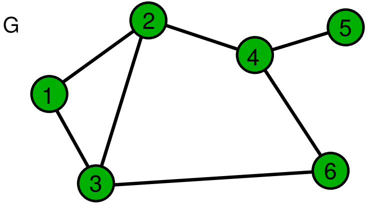
Graph Power G^k example



Adjacency matrix powers

$$A^1 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

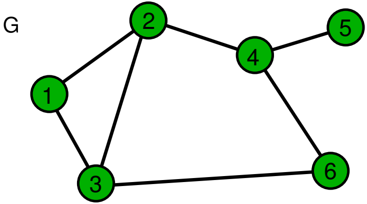
Graph Power G^k example



Adjacency matrix powers

$$A^2 = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 1 \\ 1 & 3 & 1 & 0 & 1 & 2 \\ 1 & 1 & 3 & 2 & 0 & 0 \\ 1 & 0 & 2 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 2 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Graph Power G^k example



Adjacency matrix powers

$$A^3 = \begin{pmatrix} 2 & 4 & 4 & 2 & 1 & 2 \\ 4 & 2 & 6 & 6 & 0 & 1 \\ 4 & 6 & 2 & 1 & 2 & 5 \\ 2 & 6 & 1 & 0 & 3 & 5 \\ 1 & 0 & 2 & 3 & 0 & 0 \\ 2 & 1 & 5 & 5 & 0 & 0 \end{pmatrix}$$

Graph-Power Adjacency Matrix

- To understand the above, count the number of a length 2 path between nodes i and j
- Such a path goes through an intermediate node $k \neq i, j$
- Hence the number of length two paths is

$$
\begin{aligned}
B_{ij} &= \sum_{k \neq i,j} A_{ik} A_{kj} \\
&= \sum_{k} A_{ik} A_{kj} \quad \text{because } A_{ii} = A_{jj} = 0
\end{aligned}
$$

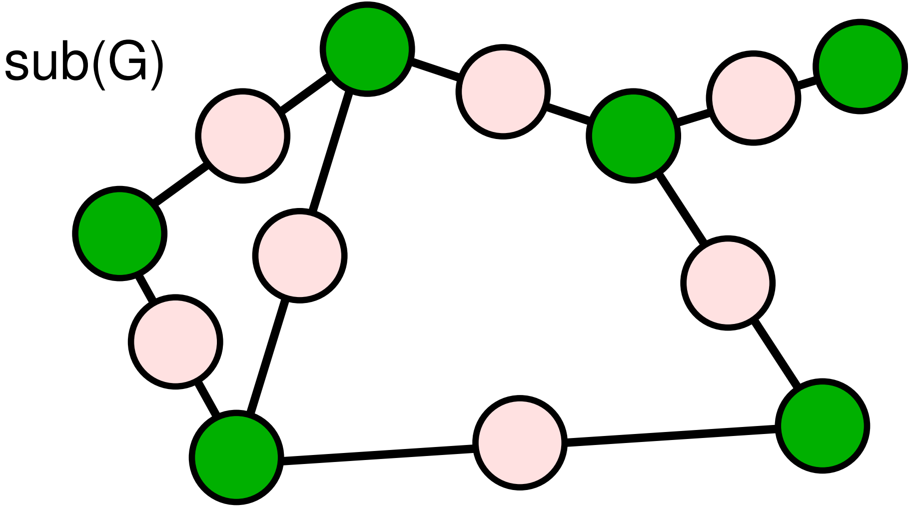
- By definition $B = A^2$
- Induction extends the argument to length k paths.

Graph-Power Properties

- For a (strongly) connected (di)graph G with n nodes, is G^n is a complete graph (or clique)?
- If the graph has diameter d, then G^d is complete.
- For an unconnected graph, the nth power will be a block-diagonal matrix whose blocks are formed by connected components.
- Square-root graph $G^{1/2}$ is a graph H such that $H^2 = G$.
- NOTE: $G^2 \neq G \times G$
  - we will talk about multiplication in the next lecture

Subdivision

- Add an extra node into an edge e



Further reading I