Complex-Network Modelling and Inference

Lecture 20: Path algebras

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Section 1

Matrix version of shortest paths

Matrix Version

We can rewrite shortest paths as the solution in the form find A^* where A^*_{ij} is the shortest-path distance between i and j and then

$$A^*_{ij} = \min_{p \in P_{ij}} w(p) = \min_{p \in P_{ij}} \sum_{e \in p} w_e,$$

where

- P_{ij} is the set of paths from i to j
- $w(p)$ is the total length of path p
- w_e is the length (or weight) of edge e

Min-plus intro

- Define new operations

$$
\begin{aligned}
a \oplus b &= \min(a, b) \\
a \otimes b &= a + b
\end{aligned}
$$

- Redefine matrix multiplication $C = A \otimes B$

Normal $C = AB$	New version $C = A \otimes B$
$C_{ij} = \sum_k A_{ik} \times B_{kj}$	$C_{ij} = \bigoplus_k A_{ik} \otimes B_{kj}$

- The new version means

$$C_{ij} = \min_k \left(A_{ik} + B_{kj} \right)$$

- We can redefine matrix *powers*

$$A^k = A \otimes A \otimes \cdots \otimes A = A \otimes A^{k-1}$$

Generalising the weighted adjacency matrix

- A is a weighted adjacency matrix

$$A_{ij} = \begin{cases} w_{ij}, & \text{if } (i,j) \in E, \\ \infty & \text{otherwise.} \end{cases}$$

  Notice ∞ instead of 0 in off-diagonal non-adjacencies

- Now A^2 using the new operators is not the number of two-hop paths, it is

$$A^2 = \bigoplus_k A_{ik} \otimes A_{kj} = \min_k \left(A_{ik} + A_{kj} \right)$$

  which is the length of the *shortest 2-hop path* (where we allow self-loops of zero length)

The meaning of matrix powers

- With the new operators we define A^k, whose elements give the shortest k-hop distances
- We have a special identity matrix I for the new operators
  - definition

$$A \otimes I = I \otimes A = A$$

  - matrix which satisfies this is

$$I = \begin{pmatrix} 0 & \infty & \infty & \dots \\ \infty & 0 & \infty & \dots \\ \infty & \infty & 0 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

  - for consistency we want $A^0 = I$, which means the length of 0-hop paths, so the definition above makes sense

Matrix Version

- The shortest path distances are then

$$A^* = min(I, A, A^2, A^3, \ldots)$$

  where I is a special identity matrix for our new operators
- We can write this as

$$A^* = I \oplus A \oplus A^2 \oplus \cdots = \bigoplus_{k=0}^{\infty} A^k$$

  - But does this sum converge?
  - How would we find it without all this work?

Matrix Version

- In normal matrix algebra

$$\begin{aligned} I + AA^* &= I + A(I + A + A^2 + \cdots) \\ &= I + A + A^2 + A^3 \cdots \\ &= A^* \end{aligned}$$

- For new operators: $\oplus$ commutes and $\otimes$ distributes over $\oplus$

$$A^* = \left(A \otimes A^*\right) \oplus I$$

So another way to think about finding A^* is to look for a solution to this equation.

  - When does one exist?
  - Is it unique?

Bellman-Ford algorithm

- We want to solve
$$A^* = \left(A \otimes A^*\right) \oplus I$$
- One approach is successive iteration
$$A^{<k+1>} = \left(A \otimes A^{<k>}\right) \oplus I$$
Hopefully it converges to a *fixed-point*, *i.e.*, the solution
- Writing this out in full, for $i \neq j$
$$A_{ij}^{<k+1>} = \min_m \left(A_{im}^{<k>} + A_{mj}^{<k>}\right)$$
This isn't Floyd-Warshall, but you can see the similarities, *e.g.*, FW recursion is
$$D_{ij}^{(k)} = \min\{D_{ij}^{(k-1)}, D_{ik}^{(k-1)} + D_{kj}^{(k-1)}\}$$
- Idea is the same: shortest-paths are built from shortest paths, but the new approach is called Bellman-Ford

Bellman-Ford algorithm

- The above is not the usual definition of Bellman-Ford
  - usually described in terms of dynamic programming
- Implementation in the Internet is *distributed* and *asynchronous* and still works!
  - there are a couple of tweaks needed
  - but its a robust, scalable approach
- The description above is nice because it generalises

Section 2

General path problems

General path problems

There are many path problems other than shortest-paths

- *connectivity*: find if a path exists
- *widest paths*: find the path with the widest “bottleneck” link
- *path reliability*: find the most reliable path
- *path security*: find the properties of the set of all possible paths

We can tackle all of these (and more) by generalising the previous matrix algebra operations $\oplus$ and $\otimes$, but we have to do so to preserve important properties – you saw that, for instance we needed:

- commutativity of $\oplus$
- distributivity
- identity

what else?

Semirings [?]

- A semiring[1] is a set S closed under 2 binary operators such that
- $(S, \oplus)$ is a commutative monoid[2] with identity $\bar{0}$
  - $\oplus$ is associative $(a \oplus b) \oplus c = a \oplus (b \oplus c)$
  - $\oplus$ commutes: $a \oplus b = b \oplus a$
  - $\oplus$ has identity $\bar{0}$: $a \oplus \bar{0} = \bar{0} \oplus a = a$
- $(S, \otimes)$ is a monoid with identity $\bar{1}$
  - $\otimes$ is associative: $(a \otimes b) \otimes c = a \otimes (b \otimes c)$
  - $\otimes$ has identity $\bar{1}$: $a \otimes \bar{1} = \bar{1} \otimes a = a$
- Multiplication distributes over addition (left and right)
  - $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$
  - $(b \oplus c) \otimes a = (b \otimes a) \oplus (c \otimes a)$
- Multiplication by $\bar{0}$ annihilates
  - $\bar{0} \otimes a = a \otimes \bar{0} = \bar{0}$

[1] Some definitions vary

[2] A monoid is a semigroup with an identity

Example Semirings $(S, \oplus, \otimes, \bar{0}, \bar{1})$

Name	S	$\oplus$	$\otimes$	$\bar{0}$	$\bar{1}$	Graph problem
Real Field	$\mathbb{R}$	$+$	$\times$	0	1	
Boolean	$\{F, T\}$	OR	AND	F	T	Reachability
(Min-+) Tropical	$\mathbb{Z}^+ \cup \infty$	min	$+$	∞	0	Shortest paths
Viterbi	$[0, 1]$	max	$\times$	0	1	Most probable path (e.g., HMMs)
Bottleneck	$\mathbb{R} \cup \pm\infty$	max	min	$-\infty$	∞	Bottleneck paths

- S is the set we work on
- $\oplus$ and $\otimes$ replace $+$ and $\times$
- $\bar{0}$ is the identity for $\oplus$
- $\bar{1}$ is the identity for $\otimes$

Less obvious examples

S	$\oplus$	$\otimes$	$\bar{0}$	$\bar{1}$	Graph problem
$\mathbb{R} \cup -\infty$	max	$+$	$-\infty$	0	Longest paths
$\mathcal{P}\{\Omega\}$	$\cup$	$\cap$	ϕ	Ω	Path properties
$\mathcal{P}\{\Omega^*\}$	$\cup$	concat	ϕ	λ	List all paths

- Ω is an arbitrary set of "symbols"
- $\mathcal{P}\{\Omega\}$ is the powerset, *i.e.*, the set of all subsets of Ω
- Ω^* is the set of all finite sequences of symbols from Ω
- λ is the empty sequence

Other operator properties

Given a set and operator $(S, \bullet)$ there are other interesting properties

selective means

$$\forall a, b \in S, \quad a \bullet b \in \{a, b\}$$

- *i.e.*, the operator "selects" one of the inputs
- *e.g.*, MIN, MAX
- *e.g.*, $\vee$ and $\wedge$
- *e.g.*, LEFT where we define

$$a \text{ left } b = a$$

idempotent means

$$\forall a \in S, \quad a \bullet a = a$$

- *i.e.*, the operator applied to the input twice does nothing
- Note *selectivity* implies *idempotence*
  Hence, *e.g.*, MIN, MAX, LEFT are idempotent
- *e.g.*, $\cup$ and $\cap$

Min-plus Semiring

The Min-plus (or Tropical) semiring defined above has

$$(S, \oplus, \otimes, \bar{0}, \bar{1}) = (\mathbb{R}, \min, +, \infty, 0)$$

Note that

- The zero element $\bar{0} = \infty$, because

$$\min(\infty, a) = \min(a, \infty) = a, \quad \forall a \in \mathbb{R}$$

  so ∞ is the "additive" identity
- The multiplicative identity $\bar{1} = 0$, because

$$0 + a = a + 0 = a \;\; \forall a \in \mathbb{R}$$

- So the *ordering* in this semiring is the opposite to what you are used, *i.e.,*
  - ∞ is small, or "bad"
  - 0 is big, or "good"

How to use the semiring

- Remember that the min-plus operators formed the basis for shortest-paths
- Other semirings form the basis for other path algebras
  - we need to choose the right semiring
  - extend it to its matrix version

Min-plus Semiring, Mark II

- Find the shortest-hop path, but only as long as it has length less than 6 hops, otherwise, treat it as invalid
- Semiring is

$$(S, \oplus, \otimes, \bar{0}, \bar{1}) = \big(\{0, 1, 2, 3, 4, 5, \infty\}, \min, \text{``}+\text{''}, \infty, 0\big)$$

where

$$a \otimes b = a \text{``}+\text{''} b = \begin{cases} a + b & \text{if } a + b < 5 \\ \infty & \text{if } a + b \geq 6 \end{cases}$$

Matrices over a Semiring form a Semiring [?]

Take $M_n(S)$ to be the set of square $n \times n$ matrices, with elements from a semiring $(S, \oplus, \otimes, \bar{0}, \bar{1})$, then we get a new semiring

$$\left(M_n(S), \hat{\oplus}, \hat{\otimes}, 0, \mathsf{I} \right)$$

- $A \hat{\oplus} B$ is element-wise addition

$$\left[A \hat{\oplus} B \right]_{ij} = a_{ij} \oplus b_{ij}$$

- $A \hat{\otimes} B$ is the generalisation of standard matrix multiplication

$$\left[A \hat{\otimes} B \right]_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes b_{kj}$$

- Identities are the same generalisation, e.g.,

$$0 = \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix}, \quad \mathsf{I} = \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{bmatrix}$$

where $\bar{1}$ and $\bar{0}$ are the identities for S

Generalised Adjacency Matrix

When working on graphs:

- give each edge a *weight*, which is an element of a S from our semiring $(S, \oplus, \otimes, \bar{0}, \bar{1})$
- describe the graph by a *generalised adjacency matrix* A where $A_{ij} \in S$ and

$$A_{ij} = \begin{cases} s_{ij} \in S, & \text{if } (i,j) \in E \\ \bar{0}, & \text{otherwise} \end{cases}$$

  where here $\bar{0}$ is the additive identity of $(S, \oplus, \otimes, \bar{0}, \bar{1})$
- These are matrices over a semiring, and so the generalised adjacency matrices also form a semiring

Graph algorithms generalise [?, ?]

Now most graph problems can be written using this model

- Our specification from before works

$$A^* = I \oplus A \oplus A^2 \oplus \cdots$$

  - remember powers in terms of $\otimes$, e.g., $A^2 = A \otimes A$
- There are more efficient algorithms
  - Floyd-Warshall is $O(n^3)$ for a network with n nodes
  - Bellman-Ford
  - Dijkstra

Reachability/connectivity [?]

Simplest example on a graph is connectivity

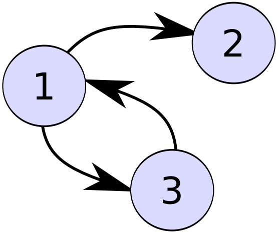
- use the Boolean semiring

$$(S, \oplus, \otimes, \bar{0}, \bar{1}) = \big(\{T, F\}, \vee, \wedge, F, T\big)$$

- $[A^k]_{ij} = T$ means, there is a path of exactly length k from i to j
  - longest path is length n for network with n nodes
- $[A^*]_{ij} = T$ means there is a path between i and j

Reachability Example

$$A = \begin{pmatrix} F & T & T \\ F & F & F \\ T & F & F \end{pmatrix}$$



where

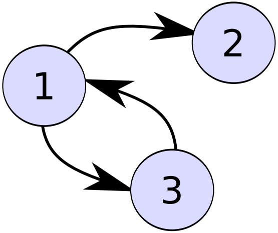
$$A_{ij} = \begin{cases} T, & \text{if } (i,j) \in E \\ F, & \text{if } (i,j) \notin E \end{cases}$$

Note $A_{ii} = F$

Reachability Example

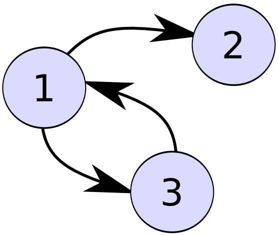
$$A = \begin{pmatrix} F & T & T \\ F & F & F \\ T & F & F \end{pmatrix}$$

$$A^2 = A \hat{\otimes} A = \begin{pmatrix} T & F & F \\ F & F & F \\ F & T & T \end{pmatrix}$$



$$[A^2]_{ij} = \begin{cases} T, & \text{if a path of length 2 exists from } i \text{ to } j \\ F, & \text{otherwise} \end{cases}$$

Reachability Example



$$A = \begin{pmatrix} F & T & T \\ F & F & F \\ T & F & F \end{pmatrix}$$

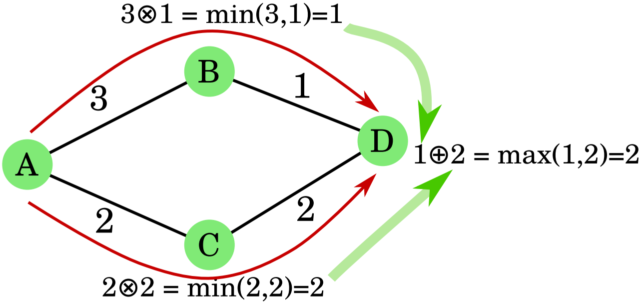
$$A^* = I \oplus A \oplus A^2 \oplus A^3 = \begin{pmatrix} T & T & T \\ F & T & F \\ T & T & T \end{pmatrix}$$

$$[A^*]_{ij} = \begin{cases} T, & \text{if a path exists from } i \text{ to } j \\ F, & \text{otherwise} \end{cases}$$

Intuition of semirings on graphs

Bottleneck Semiring Example

- $\otimes$ extends paths *in series*
- $\oplus$ combines paths *in parallel*



- result tells us the widest-bottleneck path from $A \to D$

Further reading I

Michel Gondran and Michel Minoux, *Graphs, dioids and semirings: New models and algorithms (operations research/computer science interfaces series)*, 1st ed., Springer Publishing Company, Incorporated, 2008.