ADELAIDE UNIVERSITY AUSTRALIA

Examination in School of Mathematical Sciences

Semester 2, 2007

9694,3848	TRANSFORM METHODS AND SIGNAL PROCESSING
	APP MATH 4043 and 7011

Official Reading Time:	10 mins
Writing Time:	180 mins
Total Duration:	190 mins

NUMBER OF QUESTIONS: 5 TOTAL MARKS: 100

Instructions

- Answer ALL questions.
- Begin each answer on a new page.
- Examination materials must not be removed from the examination room.

Materials

- 1 Blue books are provided.
- Calculators ARE permitted.
- Open book examination.

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO

1. (a) State the definition of the continuous Fourier transform, and then
   1. Derive the continuous Fourier transform of a pulse defined by
      $$f(t) = r(t)[\alpha + (1-\alpha)\cos(2\pi t)],$$
      where $r(t)$ is the rectangular pulse defined in lectures.
   2. Assuming $f(t)$ is used as a window function, explain two commonly chosen values of α, and why each might be used.
   3. We wish to derive a new window function based on a Gaussian function. One problem with the Gaussian window is that a Gaussian function has infinite support, and so the window function must itself be truncated. Typically this is implicit truncation by a rectangular window. However, such truncation introduces a discontinuity, and we wish to have a smoother window function, and so we will truncate our Gaussian with a triangular window function, i.e. the new window function will be
      $$f(t) = z(t) \times g_\alpha(t).$$
      where $z(t)$ is a triangular pulse of unit width, and $g_\alpha(t) = \exp(-\alpha\pi t^2)$.
      (a) Describe the Fourier transform of $f(t)$ in terms of simple functions.
      (b) Estimate the side-lobes, width, and roll-off of the new window function in comparison to the standard Gaussian window (numerical results are not required).

   (b) State the definition of the discrete Fourier transform, and then calculate DFTs of the following
   1. $(0, 1, 0, 2)$
   2. $(0, 0, 1, 0, 0, 0, 2, 0)$

   (c) Give the Haar wavelet coefficients on the dyadic grid for a signal $(0, 0, 0, 0, 1, 1, 1, 1)$ at octaves $j = 1, 2$ and 3.

[20 marks]

Solutions:

(a) [**4 marks**] $F(s) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi st}\, dt$

1. First look at two components: $f_1(t) = r(t)$ and $f_2(t) = r(t)\cos(2\pi t)$. We know that $\mathcal{F}\{f_1\} = \text{sinc}(s)$, and there are two simple approaches to calculating $\mathcal{F}\{f_2\}$, either by
   - Noting that it is a product in the time domain, and hence a convolution in the frequency domain, i.e.
     $$\mathcal{F}\{r(t)\cos(2\pi t)\} = \mathcal{F}\{r\} * \mathcal{F}\{\cos\} = \text{sinc} * \mathcal{F}\{\cos\}$$
     Now the FT of cos is $\mathcal{F}\{\cos(2\pi s_0 t)\} = \frac{1}{2}[\delta(s - s_0) + \delta(s + s_0)]$, and $G(s) * \delta(s - s_0) = G(s_0)$, so
     $$\mathcal{F}\{r(t)\cos(2\pi t)\} = \frac{1}{2}[\text{sinc}(s-1) + \sin(s+1)]$$

- A simpler method is to note the modulation property of the continuous FT, i.e., $\mathcal{F}\{g(t)\cos(2\pi s_0 t)\} = \frac{1}{2}[G(s-s_0) + G(s+s_0)]$, which leads directly to the same result.

Now, by linearity

$$\mathcal{F}\{f\} = \alpha\mathcal{F}\{f_1\} + (1-\alpha)\mathcal{F}\{f_2\} = \alpha \text{sinc}(s) + \frac{(1-\alpha)}{2}[\text{sinc}(s-1) + \sin(s+1)].$$

2. [**4 marks**] There are two values of α we have discussed in lectures corresponding to two common window functions:
   - Hanning window: $\alpha = 0.5$. The Hanning window's largest side-lobes are -31dB (with respect to the peak). As a result it considerably reduces leakage in comparison to the default rectangular window (with side lobes at -13dB). It has the disadvantage (with respect to the rectangular window) of having a wider main peak, and hence poorer resolution in the frequency domain.
   - Hamming window: $\alpha = 0.54$. The Hamming window uses the value of α that minimises the size of the first side-lobe, thereby improving the sensitivity of the window to -43dB, while also reducing the width of the main peak (with respect to the Hanning window). However, the window function now has a discontinuity, and hence the side-lobes further from the peak don't fall off as quickly as those for the Hanning window.
3. [**4 marks**] Given the function $f(t) = z(t) \times g_\alpha(t)$ where $z(t)$ is a triangular pulse of unit width, and $g_\alpha(t) = \exp(-\alpha\pi t^2)$.

   (a) To calculate the FT start with its components, and calculate the FT of these. Note that a triangular pulse (or unit width) can be constructed by a convolution of two rectangular pulses with a scaling of time to give unit width, i.e.,

   $$z(t) = [r * r](2t),$$

   Thus its Fourier transform (using the convolution and scaling properties) will be

   $$Z(s) = \frac{1}{2}\text{sinc}^2(s/2).$$

   The FT of a Gaussian $g_1(t)$ is $g_1(s)$, and so again applying the scaling property, we get

   $$G_\alpha(s) = \frac{1}{\alpha}\exp(-\pi s^2/\alpha).$$

   Applying the dual convolution property we get the FT of the complete window function

   $$\mathcal{F}\{f\} = [Z * G_\alpha](s) = \frac{1}{2\alpha}\text{sinc}^2(s/2) * \exp(-\pi s^2/\alpha).$$

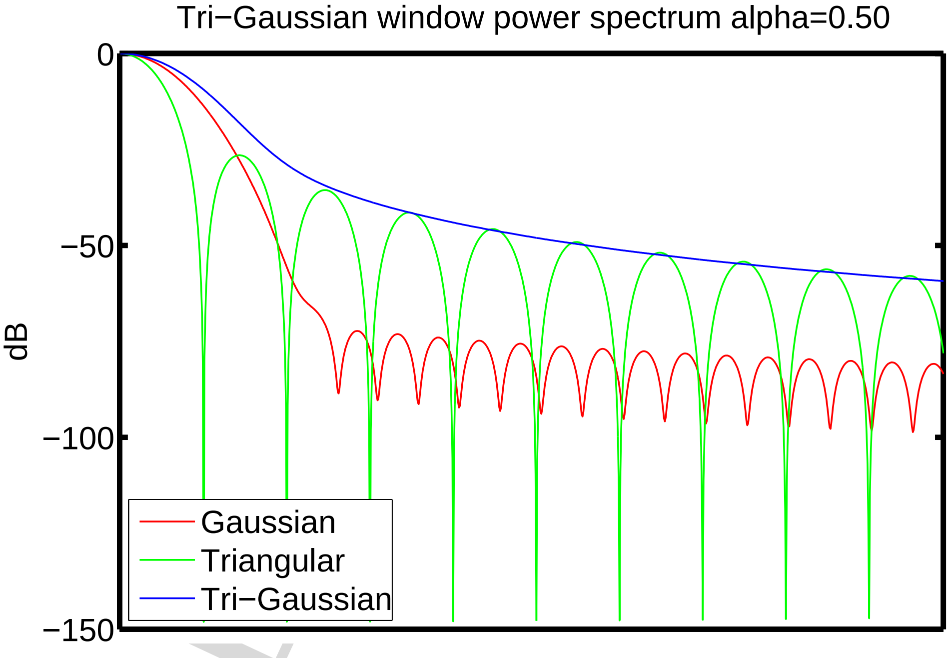
   (b) It would be somewhat painful to calculate this convolution in full, but we can guess the effect. Note that the FT of the triangular pulse is convolved with a Gaussian function as if we were filtering the Fourier transform with a Gaussian filter. Its as if the FT of the new window function is a smoothed version of that of the triangular window function. The wider the Gaussian window in the time domain (the smaller α) the less smoothing happens (the more the window looks like a triangle), and the more the window's FT looks like that for a triangular pulse. A narrower Gaussian (larger α), will produce something in the frequency domain close to a smooth curve along the peaks of those of the triangular pulse.

We can interpret this by noting that although the triangular pulse reduces the discontinuity in the Gaussian window (at its edges), it introduces a much larger discontinuity in its second derivative (at the middle), and this is decisive in determining the properties of the window function. In particular the side-lobes (where formal side-lobes exist) are (very) approximately like those of the triangular window.

The function width will vary depending on the width of the smoothing Gaussian, i.e. the value of α, but in multiplying the window function with a sharper window (the triangle instead of the rectangle) in the time domain, we might expect the window function in the frequency domain to be a little wider.

The new window function is continuous, but with a discontinuity in the 1st derivative, so it will have 6dB per octave roll-off (i.e. more roll off than the simple Gaussian window).

The following figure illustrates the three window functions for $\alpha = 0.5$.



(b) [**4 marks**]

1. We can calculate this directly by using the definition of the DFT

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-i2\pi kn/N}$$

as follows

$$\begin{array}{rclclcl} X(0) & = & e^{-i2\pi 0/4} + 2e^{-i2\pi 0/4} & = & 1+2=3 & & \\ X(1) & = & e^{-i2\pi 1/4} + 2e^{-i2\pi 3/4} & = & e^{-i\pi 1/2} + 2e^{-i\pi 3/2} & = & -i+2i=i \\ X(2) & = & e^{-i2\pi 2/4} + 2e^{-i2\pi 6/4} & = & e^{-i\pi} + 2e^{-i3\pi} & = & -1-2=-3 \\ X(3) & = & e^{-i2\pi 3/4} + 2e^{-i2\pi 9/4} & = & e^{-i\pi 3/2} + 2e^{-i\pi 9/2} & = & i-2i=-i \end{array}$$

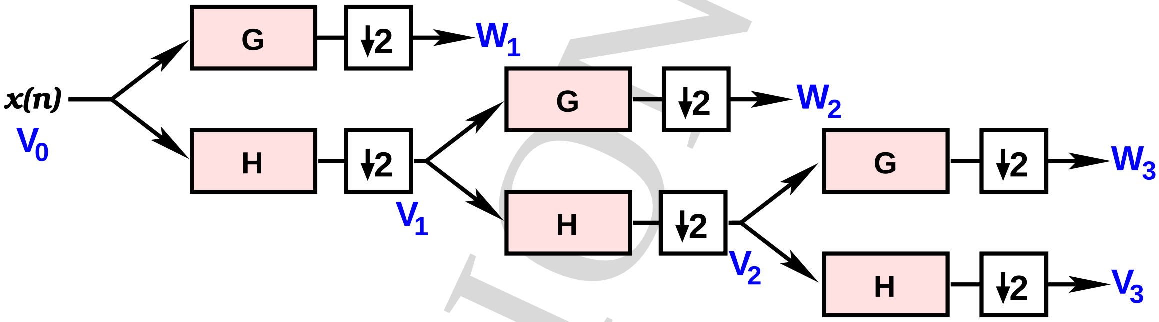
and the result is

$$\mathcal{F}\{(0,1,0,2)\} = (3, i, -3, -i)$$

2. This series is simple that from part 1.(b).i, upsampled by a factor of two. In upsampling we simply repeat the spectrum two times, to get

$$\mathcal{F}\{(0,0,1,0,0,0,2,0)\} = (3, i, -3, -i, 3, i, -3, -i)$$

(c) [**4 marks**] The discrete filters for the Haar wavelet transform are $h = (1,1)/\sqrt{2}$, and $g = (1,-1)/\sqrt{2}$, but for the purpose of simplifying the computations, we shall not use the scaling factor of $\sqrt{2}$ until the end. The pyramidal wavelet algorithm looks like



Each block refers to the use of one of the filters defined above followed by a downsampling, and each filtering corresponds to a convolution, for instance, the two first filters result in

$$\begin{aligned} x * g \downarrow 2 &= \sum_{m=-\infty}^{\infty} h(m-2n)x(m) \\ x * h \downarrow 2 &= \sum_{m=-\infty}^{\infty} g(m-2n)x(m) \end{aligned}$$

The first term is the wavelet details $d_1(n)$ (at octave $j = 1$), and the second term is the approximation $a_1(n)$ (at octave $j = 1$). So we write

$$\begin{aligned} a_1(n) &= \sum_{m=-\infty}^{\infty} h(m-2n)x(m) \\ d_1(n) &= \sum_{m=-\infty}^{\infty} g(m-2n)x(m) \end{aligned}$$

Now (ignoring the scale factors) $h(0) = 1$ and $h(1) = 1$, so

$$\begin{aligned} a_1(0) &= h(0)x(0) + h(1)x(1) \\ &= x(0) + x(1) \\ &= 0 \\ a_1(1) &= h(0)x(2) + h(1)x(3) = 0 \\ a_1(2) &= h(0)x(4) + h(1)x(5) = 2 \\ a_1(3) &= h(0)x(6) + h(1)x(7) = 2 \end{aligned}$$

So $a_1 = (0, 0, 2, 2)$. Notice there are half as many terms as the original signal because we have downsampled by 2. Similarly $g(0) = 1$ and $g(1) = -1$ so

$$\begin{aligned} d_1(0) &= g(0)x(0) + g(1)x(1) \\ &= x(0) - x(1) \\ &= 0 \\ d_1(1) &= g(0)x(2) + g(1)x(3) = 0 \\ d_1(2) &= g(0)x(4) + g(1)x(5) = 0 \\ d_1(3) &= g(0)x(6) + g(1)x(7) = 0 \end{aligned}$$

So $d_1 = (0, 0, 0, 0)$, so the approximation at octave 1 is perfect (which we could see from the signal).

Given the pyramidal structure, we can then write the following relationships between each of the larger octaves.

$$\begin{aligned} a_{j+1}(n) &= \sum_{m=-\infty}^{\infty} h(m-2n)a_j(m) = \left[a_j * \bar{h}\right](2n) \\ d_{j+1}(n) &= \sum_{m=-\infty}^{\infty} g(m-2n)a_j(m) = \left[a_j * \bar{g}\right](2n) \end{aligned}$$

If we use this to compute the second scale approximation we get

$$\begin{aligned} a_2(0) &= h(0)a_1(0) + h(1)a_1(1) \\ &= a_1(0) + a_1(1) \\ &= 0 \\ a_2(1) &= h(0)a_1(2) + h(1)a_1(3) = 4 \end{aligned}$$

So $a_2 = (0, 4)$. Likewise to compute the d_2 we get

$$\begin{aligned} d_2(0) &= g(0)a_1(0) + g(1)a_1(1) \\ &= a_1(0) - a_1(1) \\ &= 0 \\ d_2(1) &= g(0)a_1(2) + g(1)a_1(3) = 0 \end{aligned}$$

So $d_2 = (0, 0)$. We repeat to get the next higher scale.

$$\begin{aligned} a_3(0) &= h(0)a_2(0) + h(1)a_2(1) \\ &= a_2(0) + a_2(1) \\ &= 4 \\ d_3(0) &= g(0)a_2(0) + g(1)a_2(1) \\ &= a_2(0) - a_2(1) \\ &= -4 \end{aligned}$$

The MRA up to octave 3 (based on the Haar wavelets) including the $\sqrt{2}$ factors is therefore given by $\{a_3, d_1, d_2, d_3\}$ where these are

$$\begin{aligned} a_3 &= (4)/2^{3/2} \\ &= (\sqrt{2}) \\ d_3 &= (-4)/2^{3/2} \\ &= (-\sqrt{2}) \\ d_2 &= (0, 0)/2 \\ &= (0, 0) \\ d_1 &= (0, 0, 0, 0)/\sqrt{2} \\ &= (0, 0, 0, 0) \end{aligned}$$

2. (a) A company has proposed a new music encoding standard. It uses a sampling rate of 58.8kHz, at 24 bits per sample.

   1. Calculate the data rate of the encoding in kilobytes per second. How much data would we need to record a 5 minute song.
   2. Calculate the dynamic range of the new standard. Comment (briefly) on the utility of this dynamic range for encoding pop music.
   3. What is the highest frequency signal we can record without aliasing (assuming the smallest frequency signal present will have frequency 0)?
   4. Given that most people can't hear frequencies above 20kHz, why would we want to record music at 58.8kHz (in comparison say to the standard CD sample rate of 44.1kHz).

   (b) High capacity submarine cables are rather expensive, and so only get build in certain places. For other international links the typical approach is to use a satellite connection. Given limited satellite capacity (say between Australia and New Guinea), it would be desirable to compress the above signal such that its bit rate is substantially lower. Briefly describe one approach to compression that we have discussed in lectures (your answer should take no more than half a page of text, though you may supplement this with diagrams). Make sure that you explain the intuition behind the approach.

   (c) Given an initial recording sampled at 58.8kHz, we still need to create a version that we can write to a standard CD for the mass market. Describe the required steps to create such a recording.

[20 marks]

Solutions:

(a) [**8 marks**] A company has proposed a new music encoding standard. It uses a sampling rate of 58.8kHz, at 24 bits per sample.

1. The bit rate is just $24 \times 58,800 = 1,411,200$ b/s. Now 5 minutes = 300 seconds, so the total is $423,600,000$ bits, or 52.92 Mbytes.
2. The dynamic range will be approximately $23 \times 6 = 138$ dB (ignoring the sign bit), or 144dB including the sign bit. We could work this out more accurately, but it isn't needed here because this is far more than a CD, and most pop music already utilises only a small proportion of the dynamic range available on a CD (RE: see handout on dynamic range compression "The future of music").
3. The highest frequency will be half the sampling rate, i.e., 29.4 kHz, though in practice some guard band should be included around this.
4. There are two reasons that have been discussed (bonus marks for those who get both)
   - **the influence of higher frequencies on timbre:** The sampling theorems typically assume a stationary signal, but real signals have transients. Perhaps some detail is lost in these transients on sampling. An alternative view is that higher harmonics of instruments may not be audible, but perhaps still influence the timbre of instruments. Also, interactions of instruments higher harmonics may influence the

overall feel of music when they are mixed (but not if they have been pre-filtered out of the signal).

- **quality of analogue filters:** It is hard to build cheap, reliable analogue filters with good stop band-attenuation and sharp transition regions. If we sample at a higher rate, we can bandlimit the input signal with a simpler, cheaper analogue filter that has a wider transition region, while still preventing aliasing.

For both of these reasons it might make sense to record the original sound signals with a higher sampling rate than we finally use in the version produced for the mass market. A third reason sometimes cited is stereo imaging. Our ability to image (i.e. place things in 3D) using stereo sound depends on resolving time-difference of arrival measurements down to very small intervals, perhaps smaller than the sample intervals on a standard CD.

(b) [**8 marks**] Compression involves reducing the memory requirements for a set of data (e.g. our recorded signal). There are two types of compression: lossless, and lossy, but only lossy compression is considered in this course. The aim off lossy compression is to reduce the number of bits required to store (or transmit) the signal, but with minimal *perceivable* change to the signal. For example, with our voice signal we would like to encode the signal with fewer bits, but with few audible artifacts. The key idea in many such compression schemes is that errors introduced in the frequency domain are often less easily perceived than errors in the time-domain. Hence, a typical approach to compression is to transform the signal (e.g. using a cosine transform), and quantise the signal (more coarsely) in the frequency domain. We are less sensitive to errors in certain parts of the spectrum, and so these parts can be quantised with fewer bits.

[*extra credit:* In audio signals, a phenomena called masking occurs. A loud signal in one frequency band can hide a small signal in another band, and so we can quantise adaptively in the frequency domain to take advantage of any masking that occurs].

[*side note:* The other form of compression (e.g. dynamic range compression) might also be used to reduce bit rates, but with a much greater loss in fidelity through the introduction of quantization noise, and so we would not consider this approach here).]

(c) [**4 marks**] Changing the sampling rate, or "resampling" can be performed in a number of ways, but the method discussed in class is as follows.

1. Upsample to least-common-multiple (LCM) of the two sampling rates: in this case the two sampling rates are 44.1 and 58.8kHz, so we need to upsampling to 176.4kHz (upsampling the 58.8kHz signal by a factor of 3). Upsampling can be accomplished by similarity property, i.e., if we want to upsample by a factor of p, we insert $p-1$ zeros between each sample. The result will be a spectrum which has p copies of the original spectrum, so we then need to low-pass this spectrum to select on the part equivalent to our original spectrum
2. Downsample to the desired rate, in this case, downsampling by a factor of 4. Downsampling by a factor of q involves pre-filtering the signal with a low-pass to remove any frequencies above the new Nyquist frequency, and then removing $q-1$ samples so that we retain every qth sample.

The net result is a resampling by a factor of p/q which in this case is $3/4$ to the new rate of 44.1kHz.

For bonus marks note that there are clever filtering tricks we can use to make the filtering more efficient, i.e. in upsampling by filtering before the upsampling operation (so the filtering takes place on the lower rate signal), and likewise in downsampling we can place the filter after the downsampling operation. In fact, we can rearrange the whole process to minimise the number of operations needed, and if we were careful we may even involve ever having to deal with the 176.4kHz signal.

In addition the quantization must be changed from 24 bits to 16 bits. This is done using a "rounding-off" type of operation, i.e. dropping the least significant 8 bits.

3. The continuous cosine transform can be defined by

$$\mathcal{C}\{f(t); s\} = \int_{-\infty}^{\infty} f(t)\cos(2\pi st)\,dt$$

(a) Calculate the cosine transforms of

1. $\delta(t)$
2. $e^{-2\pi i s_0 t}$
3. The rectangular pulse $r(t)$.
4. $\sin(2\pi s_0 t)$

(b) Show that for a real input signal, the cosine transform is just the real part of a Fourier transform.

(c) Derive the properties of the cosine transform equivalent to the following properties of the Fourier transform.

1. linearity
2. time scaling
3. convolution [hint: you may have to define a sine transform in the same way we define the cosine transform above].
4. differentiation I

(d) Name an application where discrete cosine transform are commonly used. Why is a cosine transform used instead of the Fourier transform?

[20 marks]

Solutions: The continuous cosine transform can be defined by

$$\mathcal{C}\{f; s\} = \int_{-\infty}^{\infty} f(t)\cos(2\pi st)\,dt$$

(a) [**8 marks**] Calculate the cosine transforms of

1. By the sifting property of the Dirac delta

$$\mathcal{C}\{\delta(t); s\} = \int_{-\infty}^{\infty} \delta(t)\cos(2\pi st)\,dt = \cos(0) = 1$$

2. Note that we can rewrite this as a FT, i.e.,

$$\begin{aligned}
\mathcal{C}\left\{e^{-2\pi i s_0 t}; s\right\} &= \int_{-\infty}^{\infty} e^{-2\pi i s_0 t}\cos(2\pi st)\,dt \\
&= \mathcal{F}\{\cos(2\pi st); s_0\} \\
&= \frac{1}{2}\left[\delta(s_0 - s) + \delta(s_0 + s)\right]
\end{aligned}$$

3.

$$\begin{aligned}
\mathcal{C}\{r(t);s\} &= \int_{-\infty}^{\infty} r(t)\cos(2\pi st)\,dt \\
&= \int_{-1/2}^{1/2} \cos(2\pi st)\,dt \\
&= \left[\frac{\sin(2\pi st)}{2\pi s}\right]_{-1/2}^{1/2} \\
&= \frac{\sin(\pi s)}{\pi s} \\
&= \text{sinc}(s)
\end{aligned}$$

4. $\mathcal{C}\{\sin(2\pi s_0 t);s\} = \int_{-\infty}^{\infty} \sin(2\pi s_0 t)\cos(2\pi st)\,dt$, but note that sin is an odd function, and cos even, so the product of the two will be odd, and hence the integral evaluates to zero, i.e., $\mathcal{C}\{\sin(2\pi s_0 t);s\} = 0$ for all s.

(b) [**2 marks**]

$$\begin{aligned}
\mathcal{F}\{f(t);s\} &= \int_{-\infty}^{\infty} f(t)e^{-2\pi ist}\,dt \\
&= \int_{-\infty}^{\infty} f(t)\left[\cos(2\pi st) - i\sin(2\pi st)\right]\,dt \\
&= \int_{-\infty}^{\infty} f(t)\cos(2\pi st)\,dt - i\int_{-\infty}^{\infty} f(t)\sin(2\pi st)\,dt
\end{aligned}$$

The real part of this is just the cosine transform.

(c) [**8 marks**] Derive the properties of the cosine transform equivalent to the following properties of the Fourier transform.

1. **linearity:** the transform is an integral transform and is therefore linear by the linearity of the integral.
2. **time scaling:** the cosine transform of $f(at)$ will be given by

$$\begin{aligned}
\mathcal{C}\{f(at);s\} &= \int_{-\infty}^{\infty} f(at)\cos(2\pi st)\,dt \\
&= \frac{1}{|a|}\int_{-\infty}^{\infty} f(s)\cos(2\pi(s/a)t)\,dt \\
&= \frac{1}{|a|}\mathcal{C}\{f(t);s/a\}
\end{aligned}$$

by a simple change of variables.
3. **convolution:** There are two simple derivations: the direct way is a simple change of

the order of integration (where both integrals are finite)

$$
\begin{aligned}
\mathcal{C}\{[f * g](t)\} &= \mathcal{C}\left\{ \int_{-\infty}^{\infty} f(u)\, g(t-u)\, du \right\} \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(u)\, g(t-u)\, du \right] \cos(2\pi st)\, dt \\
&= \int_{-\infty}^{\infty} f(u) \int_{-\infty}^{\infty} g(t-u) \cos(2\pi st)\, dt\, du \\
&= \int_{-\infty}^{\infty} f(u) \int_{-\infty}^{\infty} g(t) \cos(2\pi s(t+u))\, dt\, du \\
&= \int_{-\infty}^{\infty} f(u) \int_{-\infty}^{\infty} g(t) \left[\cos(2\pi st)\cos(2\pi su) - \sin(2\pi st)\sin(2\pi su) \right] dt\, du \\
&= \int_{-\infty}^{\infty} f(u) \cos(2\pi su) \int_{-\infty}^{\infty} g(t)\cos(2\pi st)\, dt\, du \\
&\quad - \int_{-\infty}^{\infty} f(u) \sin(2\pi su) \int_{-\infty}^{\infty} g(t)\sin(2\pi st)\, dt\, du \\
&= \int_{-\infty}^{\infty} f(u)\cos(2\pi su)\mathcal{C}\{g(t); s\}\ du \\
&\quad - \int_{-\infty}^{\infty} f(u)\sin(2\pi su)\mathcal{S}\{g(t); s\}\ du \\
&= \mathcal{C}\{g(t); s\} \int_{-\infty}^{\infty} f(u)\cos(2\pi su)\, du \\
&\quad - \mathcal{S}\{g(t); s\} \int_{-\infty}^{\infty} f(u)\sin(2\pi su)\, du \\
&= \mathcal{C}\{g(t); s\}\, \mathcal{C}\{f(t); s\} - \mathcal{S}\{g(t); s\}\, \mathcal{S}\{f(t); s\}
\end{aligned}
$$

where $\mathcal{S}\{f\}$ is the sine transform defined by direct analogy to the cosine transform above.

A second approach is to note that the cosine transform is the real part of the Fourier transform, and that this will be the real part of the product of the FT of the components which (by the standard convolution theorem) is simple the product of the real parts of the FTs of these components minus the product of the complex parts (the sine transform) of the FTs of these components.

4. **differentiation I:** Note that $\mathcal{C}\{f\} = \Re(\mathcal{F}\{f\})$

$$
\mathcal{F}\{f\} = \mathcal{C}\{f\} + i\mathcal{S}\{f\}
$$

and that the differentiation property of the FT is $\mathcal{F}\left\{\frac{d^n}{dt^n} f(t)\right\} = (i2\pi s)^n \mathcal{F}\{f\}$. Taking the real part of this we get

$$
\mathcal{C}\left\{ \frac{d^n}{dt^n} f \right\} = \begin{cases} (-1)^{(n+1)/2} (2\pi s)^n \mathcal{S}\{f\} & \text{for } n \text{ odd} \\ (-1)^{n/2} (2\pi s)^n \mathcal{C}\{f\} & \text{for } n \text{ even} \end{cases}
$$

(d) [**2 marks**] Discrete cosine transforms are used in JPEG image compression. DCTs are used rather than DFTs because of the reduction on computation load by using real arithmetic (rather than the complex arithmetic used by the DFT).

Please turn over for page 15

4. Filters.

   (a) A filter's z-transform $H(z)$ has two poles at $z = 1 \pm i$ in the complex plane, and two zeros at $z = 0$.

      1. Write its z-transform as a ratio of polynomials.
      2. Draw a box diagram showing how you might implement this filter.
      3. Characterize this filter, by stating (with reasons) whether it is
         (a) linear
         (b) time-invariant
         (c) FIR or IIR
         (d) causal
         (e) stable
         (f) would you expect it to look more like a high-pass or low-pass filter.
      4. Calculate the first 12 terms of the impulse response of the filter, and relate these to the stability of the filter described above.
      5. Take a filter of the form $y(n) = ay(n-1) - ay(n-2) + x(n)$. By calculating the z-transform of this filter determine the values of a for which such a filter would be stable.

   (b) The Butterworth filter of order n has transfer function $H(s)$ with respect to frequency s that satisfies

$$|H(s)|^2 = \frac{1}{1 + (s/s_c)^{2n}}$$

      1. Describe why you might believe this filter to be a low-pass filter.
      2. Comment on the limit of this filter as $n \to \infty$.
      3. The Butterworth filter is sometimes characterised as a filter of "maximum flatness" in the pass band. Why might this be?

[20 marks]

Solutions:

(a) [**12 marks**] A filter's z-transform $H(z)$ has two poles at $z = 1 \pm i$ in the complex plane, and two zeros at $z = 0$.

   1. The z-transform will be given by

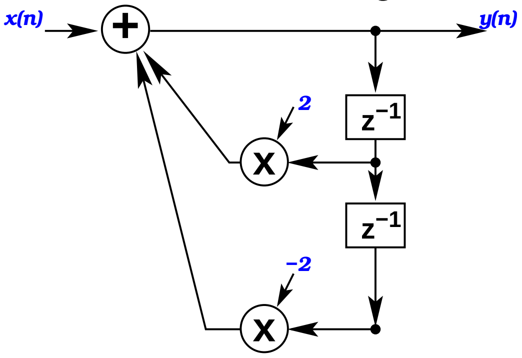
$$\begin{aligned} H(z) &= \frac{z^2}{(z-1+i)(z-1-i)} \\ &= \frac{z^2}{z^2 - 2z + 2} \\ &= \frac{1}{1 - 2z^{-1} + 2z^{-2}} \end{aligned}$$

   either of the second two forms is acceptable, but the second is more useful for the next step.

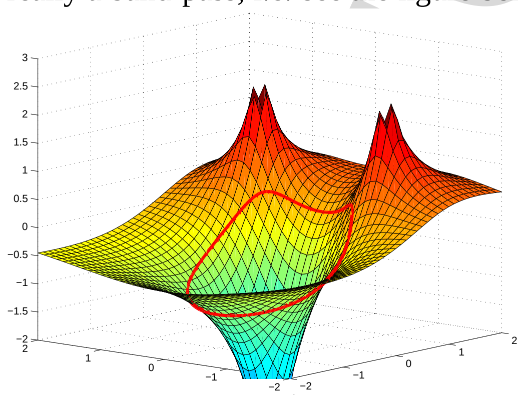
2. Taking the standard ARMA form of the filter, $B(z) = 1$, and $A(z) = 1 - 2z^{-1} + 2z^{-2}$, and we can immediately recognise the co-efficients in this such that $y(n) - 2y(n-1) + 2y(n-2) = x(n)$ or

$$y(n) = 2y(n-1) - 2y(n-2) + x(n)$$

The standard form of box diagram will then look like



3. characterise this filter, by stating (with reasons) whether it is
   (a) linear: yes, because it is a linear combination of inputs.
   (b) time-invariant: yes, as the coefficients are const WRT time.
   (c) IIR: because the AR co-efficients are non-trivial.
   (d) causal: yes, doesn't depend on future of a signal.
   (e) stable: no, poles are outside unit circle in complex plane.
   (f) poles are nearer low frequencies, so more like a low pass (though in actuality it is really a band-pass, i.e. see the figure below).



4. impulse input $x(n) = \delta_{n0}$, i.e., $x(n) = 0$ for all $n \neq 0$ and $x(0) = 1$. Assume $y(-n) = 0$ for $n > 0$, and we can calculate $y(n) = -2y(n-1) + 2y(n-2) + x(n)$

$$\begin{aligned} y(0) &= x(0) = 1 \\ y(1) &= 2y(0) + x(1) = 2 \\ y(2) &= 2y(1) - 2y(0) + x(2) = 2 \\ y(3) &= 2y(2) - 2y(1) + x(3) = 0 \end{aligned}$$

and so on to get

$$y(0, \ldots, 12) = [1, 2, 2, 0, -4, -8, -8, 0, 16, 32, 32]$$

Clearly we can see that even with this simple input (which is bounded) the output will grow in magnitude, and so is not bounded, i.e. the filter is not stable (in the BIBO sense).

5. The z-transform of this filter will be

$$H(x) = \frac{z^2}{z^2 - az + a},$$

which has two zeros at $z = 0$ and two poles at

$$z = \frac{a \pm \sqrt{a^2 - 4a}}{2}.$$

When $a \leq 0$ the poles will be real, and the one furthest from the centre of the complex plane will be the negative root $z = \frac{a-\sqrt{a^2-4a}}{2}$, and for stability we need the absolute value of this to be less than 1, so stability imposes the condition

$$\begin{aligned} \frac{a - \sqrt{a^2 - 4a}}{2} &> -1 \\ \sqrt{a^2 - 4a} &< 2 + a \\ a^2 - 4a &< 4 + 4a + a^2 \\ a &> -1/2 \end{aligned}$$

When $a \geq 4$ the poles will be real, but $a/2 \geq 2$, and so there will always be at least one pole outside the unit circle.
When $0 < a < 4$, then the poles will be complex and we can write them

$$z = \frac{a}{2} \pm \frac{\sqrt{4a - a^2}}{2} i.$$

The absolute value (squared) of these poles will be

$$|z|^2 = \frac{a^2}{4} + \frac{4a - a^2}{4} = a.$$

Hence, the condition for stability is $a < 1$, and so the filter will be stable for

$$-1/2 < a < 1.$$

(b) [**8 marks**] The Butterworth filter of order n has transfer function $H(s)$ that satisfies

$$|H(s)|^2 = \frac{1}{1 + (s/s_c)^{2n}}$$

1. For $n \geq 1$, the squared magnitude of the transfer function varies (smoothly from unity $s = 0$) to $1/2$ at the "critical frequency" s_c, and then decreases for increasing s. So it is a low-pass filter (high-pass versions can be created by reversing the role of s and s_c.).

2. As $n \to \infty$ the function

$$(s/s_c)^{2n} \to \begin{cases} 0, & \text{for } |s| < s_c \\ 1 & \text{for } |s| = s_c \\ \infty, & \text{for } |s| > s_c \end{cases}$$

   And so the transfer function as $n \to \infty$ looks like

$$|H(s)|^2 = \begin{cases} 1, & \text{for } |s| < s_c \\ 1/2 & \text{for } |s| = s_c \\ 0, & \text{for } |s| > s_c \end{cases}$$

   i.e. it looks like an ideal filter with pass-band $|s| < s_c$.

3. The derivative (of the squared magnitude) of the transfer function is

$$\frac{d}{ds}\frac{1}{1+(s/s_c)^{2n}} = \frac{-(s/s_c)^{2n-1}}{s_c(1+(s/s_c)^{2n})^2}$$

   Now, for this to be zero, we need $(s/s_c)^{2n-1} = 0$, which is only the case at $s = 0$ (for $n \geq 1$) and so the transfer function has no turning points, and hence no ripples, i.e., it is “flat”.

5. Write a brief (less than one page) essay contrasting the Short-Time Fourier Transform and the Wavelet Transform. In particular, discuss how each deals with uncertainty bounds (and why this is important), and how the construction of the wavelet basis functions is helpful in this regard. Highlight why the Wavelet transform would often be a preferable.

   Please note that approximately 50% of marks will be based on content, and 50% on presentation, including clarity of your arguments. [20 marks]

   Solutions: Obviously no correct solution exists to an essay question. In marking the question I will take into account the

   - clarity of writing, and quality of presentation,
   - the following points should be covered
     - the uncertainty principle (definition and consequences)
     - the different ways the wavelet and break up the time frequency space
     - the construction of the wavlets from a mother wavelet by dilations and translations, and the discretization to the dyadic grid which result in a set of basis functions which break up the time-frequency domain in the required manner.
     - the fact that the wavelet's break up of the time-frequency domain is in many cases more desirable because it breaks the time domain into block of an appropriate resolution for the frequency under consideration.

   Note the page limit, and stick to it.