Variational Methods & Optimal Control

lecture 10

Matthew Roughan
<matthew.roughan@adelaide.edu.au>

Discipline of Applied Mathematics
School of Mathematical Sciences
University of Adelaide

April 14, 2016

Extension 2: several dependent variables

When there are several dependent variables, i.e., y is a vector, then the E-L equations generalize to give one DE per dependent variable. A simple example is when we calculate the trajectory of a particle in 3D. This section introduces a number of physics ideas/principles: potentials, Lagrangians, Hamilton's principle, Newton's laws of motion, and conservations laws.

Extension

Several dependent variables

- in prior problem formulations, we have only one dependent variable y, which is dependent on x, e.g. $y = y(x)$.
- we can extend this to many dependent variables q_i
- a typical example might be the position of a particle in 3D space with respect to time, e.g. $(x(t), y(t), z(t))$
- the particle has three dependent variables x, y and z

Definitions

Define $\mathbf{C}^2[t_0,t_1]$ to denote the set of vector functions $\mathbf{q} : [t_0,t_1] \to I\!R^n$, such that for $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ its component functions $q_k \in C^2[t_0,t_1]$ for $k = 1, 2, \ldots, n$.

- i.e. take a set of n functions $q_k(t)$, with two continuous derivatives with respect to t, and put them into a vector $\mathbf{q}(t)$
- dot notation:

$$\dot{q}_k = \frac{dq_k}{dt}, \quad \ddot{q}_k = \frac{d^2 q_k}{dt^2} \quad \text{and} \quad \dot{\mathbf{q}} = \left(\frac{dq_1}{dt}, \frac{dq_2}{dt}, \ldots, \frac{dq_n}{dt} \right)$$

- we can define norms on the space $\mathbf{C}^2[t_0,t_1]$, e.g.

$$\|\mathbf{q}\| = \max_{k=1,\ldots,n} \sup_{t \in [t_0,t_1]} |q_k(t)|$$

Functionals

We can define functionals, for example

$$F\{\mathbf{q}\} = \int_{t_0}^{t_1} L(t, \mathbf{q}, \dot{\mathbf{q}})\, dt$$

where we choose the function L to have continuous 2nd-order derivatives with respect to t, q_k and $\dot{q}_k$, for $k = 1, \ldots, n$.

For the fixed end-point problem, we look for $\mathbf{q} \in S$, where

$$S = \{\mathbf{q} \in \mathbf{C}_2^n[t_0, t_1] | \mathbf{q}(t_0) = \mathbf{q}_0, \mathbf{q}(t_1) = \mathbf{q}_1\}$$

Extremals

As before, we look for extremals by examining perturbations of $\mathbf{q}$, and seeing their effect on the functional, e.g. take the perturbation

$$\hat{\mathbf{q}} = \mathbf{q} + \varepsilon \mathbf{n}$$

where $\mathbf{n} \in \mathcal{H}^n$, where

$$\mathcal{H} = \left\{ n_i \in \mathbf{C}^2[t_0, t_1] | n_i(t_0) = 0, n_i(t_1) = 0 \right\}$$

For instance, for a local minima, we require

$$F\{\mathbf{q} + \varepsilon \mathbf{n}\} \geq F\{\mathbf{q}\}$$

for all $\mathbf{n} \in \mathcal{H}^n$ and $\mathbf{q} + \varepsilon \mathbf{n}$ in a small neighborhood of $\mathbf{q}$ with respect to some distance metric.

Applying Taylor's theorem

Taylor's theorem (again)

$$f(\mathbf{x} + \delta \mathbf{x}) = f(\mathbf{x}) + \sum_{i=1}^{n} \delta x_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i x_j} \delta x_i \delta x_j + O(\delta \mathbf{x}^3)$$

Applying with $\mathbf{x} = (t, \mathbf{q}, \dot{\mathbf{q}})$, and $\delta \mathbf{x} = (0, \varepsilon \mathbf{n}, \varepsilon \dot{\mathbf{n}})$

$$L(t, \mathbf{q} + \varepsilon \mathbf{n}, \dot{\mathbf{q}} + \varepsilon \dot{\mathbf{n}}) = L(t, \mathbf{q}, \dot{\mathbf{q}}) + \varepsilon \sum_{k=1}^{n} \left(n_k \frac{\partial L}{\partial q_k} + \dot{n}_k \frac{\partial L}{\partial \dot{q}_k} \right) + O\left(\varepsilon^2\right)$$

Deriving the Euler-Lagrange eq.s

As before the **First Variation** is

$$\begin{aligned} \delta F(\mathbf{n}, \mathbf{q}) &= \frac{F\{\mathbf{q} + \varepsilon \mathbf{n}\} - F\{\mathbf{q}\}}{\varepsilon} \\ &= \frac{1}{\varepsilon} \int_{t_0}^{t_1} L(t, \mathbf{q} + \varepsilon \mathbf{n}, \dot{\mathbf{q}} + \varepsilon \dot{\mathbf{n}}) - L(t, \mathbf{q}, \dot{\mathbf{q}})\, dt \\ &= \int_{t_0}^{t_1} \sum_{k=1}^{n} \left(n_k \frac{\partial L}{\partial q_k} + \dot{n}_k \frac{\partial L}{\partial \dot{q}_k} \right) dt + O(\varepsilon) \\ &= 0 \end{aligned}$$

for all $\mathbf{n} \in \mathcal{H}^n$ as $\varepsilon \to 0$.

This is still a little too hard for us

Deriving the Euler-Lagrange eq.s

Note the above must be true for all $\mathbf{n} \in \mathcal{H}^n$.

We can simplify by choosing: $\mathbf{n}_1 = (n_1, 0, 0, \ldots, 0)$.

Then the First Variation simplifies

$$
\begin{aligned}
\delta F(\mathbf{n}_1, \mathbf{q}) &= \int_{t_0}^{t_1} \sum_{k=1}^{n} \left(n_k \frac{\partial L}{\partial q_k} + \dot{n}_k \frac{\partial L}{\partial \dot{q}_k} \right) dt \\
&= \int_{t_0}^{t_1} \left(n_1 \frac{\partial L}{\partial q_1} + \dot{n}_1 \frac{\partial L}{\partial \dot{q}_1} \right) dt
\end{aligned}
$$

We integrate the term $\dot{n}_1 \frac{\partial L}{\partial \dot{q}_1}$ by parts as in the derivation of the simple Euler-Lagrange equation and we get

Deriving the Euler-Lagrange eq.s

$$
\delta F(\mathbf{n}_1, \mathbf{q}) = \int_{t_0}^{t_1} n_1 \left(\frac{\partial L}{\partial q_1} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} \right) dt
$$

For an extremal we want $\delta F(\mathbf{n}_1, \mathbf{q}) = 0$
for all $n_1 \in \mathcal{H} = \{C^2[t_0, t_1] | n_1(t_0) = 0, n_1(t_1) = 0\}$
Applying Lemma 2.2.2 gives

$$
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} = 0
$$

This is directly analogous to the original Euler-Lagrange equation.

Deriving the Euler-Lagrange eq.s

We can do likewise for

$$
\mathbf{n}_k = (0, 0, \ldots, 0, n_k, 0, \ldots, 0)
$$

in exactly the same fashion to obtain a set of equations

$$
\begin{aligned}
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} &= 0 \\
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} &= 0 \\
&\vdots \\
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_n} - \frac{\partial L}{\partial q_n} &= 0
\end{aligned}
$$

The result is analogous to maximizing a function of several variables, where we must set all of the partial derivatives $\partial f / \partial x_k = 0$.

Simple example

Find extremals of

$$
F\{\mathbf{q}\} = \int_0^1 \left(\dot{q}_1^2 + (\dot{q}_2 - 1)^2 + q_1^2 + q_1 q_2 \right) dt
$$

for $\mathbf{q}(0) = \mathbf{q}_0$ and $\mathbf{q}(1) = \mathbf{q}_1$

The Euler-Lagrange equations are

$$
\begin{aligned}
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} &= 0 \\
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_2} - \frac{\partial L}{\partial q_2} &= 0
\end{aligned}
$$

Simple example

$$L = \left(\dot{q}_1^2 + (\dot{q}_2 - 1)^2 + q_1^2 + q_1 q_2\right)$$

So

$$\begin{aligned} \frac{\partial L}{\partial q_1} &= 2q_1 + q_2, & \frac{\partial L}{\partial q_2} &= q_1 \\ \frac{\partial L}{\partial \dot{q}_1} &= 2\dot{q}_1, & \frac{\partial L}{\partial \dot{q}_2} &= 2(\dot{q}_2 - 1) \end{aligned}$$

So the E-L equations are

$$\begin{aligned} 2\ddot{q}_1 - 2q_1 - q_2 &= 0 \\ 2\ddot{q}_2 - q_1 &= 0 \end{aligned}$$

Simple example

Differentiate the second equation twice with respect to t to get

$$2q_2^{(4)} - \ddot{q}_1 = 0$$

which we rearrange to get $\ddot{q}_1 = 2q_2^{(4)}$, which we can substitute (along with the second equation $q_1 = 2\ddot{q}_2$) into the first equation to get a 4th order DE for q_2, e.g.

$$4q_2^{(4)} - 4\ddot{q}_2 - q_2 = 0$$

Simple example

The forth order linear ODE

$$2q_2^{(4)} - 2\ddot{q}_2 - \frac{1}{2}q_2 = 0$$

has characteristic equation

$$2\mu^4 - 2\mu^2 - 1/2 = 0$$

which has roots

$$\begin{aligned} \mu_1, \mu_2 &= \pm\sqrt{\frac{1}{2} + \frac{1}{\sqrt{2}}} \\ \mu_3, \mu_4 &= \pm\sqrt{\frac{1}{2} - \frac{1}{\sqrt{2}}} = \pm im \end{aligned}$$

Simple example

The solution is

$$q_2(t) = c_1 e^{\mu_1 t} + c_2 e^{\mu_2 t} + c_3 \cos(mt) + c_4 \sin(mt)$$

where c_1, c_2, c_3 and c_4 are determined by the 4 end-point conditions $\mathbf{q}(0) = \mathbf{q}_0$ and $\mathbf{q}(1) = \mathbf{q}_1$.

We can determine q_1 from

$$q_1 = 2\ddot{q}_2 = 2c_1\mu_1^2 e^{\mu_1 t} + 2c_2\mu_2^2 e^{\mu_2 t} - 2c_3 m^2 \cos(mt) - 2c_4 m^2 \sin(mt)$$

Example: movement of a particle

The **kinetic energy** of a particle is

$$T = \frac{1}{2} m v^2(t) = \frac{1}{2} m \left(\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t) \right)$$

where $v(t)$ is the speed of the particle at time t.

Assume there exists a scalar function of time and position $V(t,x,y,z)$, such that the forces acting on the particle are

$$f_x = -\frac{\partial V}{\partial x},\ f_y = -\frac{\partial V}{\partial y},\ f_z = -\frac{\partial V}{\partial z}$$

Then V is called the **potential energy** of the particle.

The Lagrangian

The function $L(t,x,y,x,\dot{x},\dot{y},\dot{z})$

$$L = T - V$$

is called the **Lagrangian**

The path of a particle is given by $\mathbf{r}(t) = (x(t), y(t), z(t))$ over the time interval $[t_0, t_1]$.

We can define the **action integral** by

$$F\{\mathbf{r}\} = \int_{t_0}^{t_1} L(t, \mathbf{r}, \dot{\mathbf{r}})\, dt$$

Hamilton's principle

The path of a particle $\mathbf{r}(t)$ is such that the functional

$$F\{\mathbf{r}\} = \int_{t_0}^{t_1} L(t, \mathbf{r}, \dot{\mathbf{r}})\, dt$$

is stationary.

- could be a saddle point (not just minima)
- note, Hamilton's principle is far more general
  - multiple particles
  - non-Cartesian coordinates
  - remember changing coordinates shouldn't change extremal curves

Generalized coordinates

We can describe the mechanical system by generalized coordinates $\mathbf{q}(t)$.

- The kinetic energy is given by $T(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \sum_{j,k=1}^{n} C_{j,k}(\mathbf{q}) \dot{q}_j \dot{q}_k$
- The potential energy is given by $V(t, \mathbf{q})$
- The Lagrangian is $L(t, \mathbf{q}, \dot{\mathbf{q}}) = T(\mathbf{q}, \dot{\mathbf{q}}) - V(t, \mathbf{q})$

Hamilton's principle states that the path of the particle $\mathbf{q}(t)$ will be such that the functional

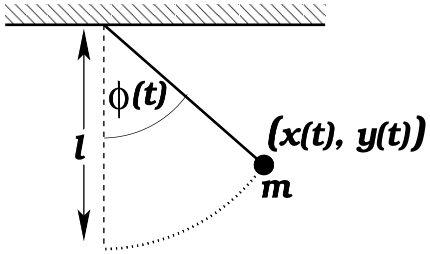
$$F\{\mathbf{q}\} = \int_{t_0}^{t_1} L(t, \mathbf{q}, \dot{\mathbf{q}})\, dt$$

is stationary.

Example: a simple pendulum

Kinetic energy

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}ml^2\dot{\phi}^2$$



Potential energy

$$V = mg(l - y) = mgl(1 - \cos\phi)$$

The Lagrangian is

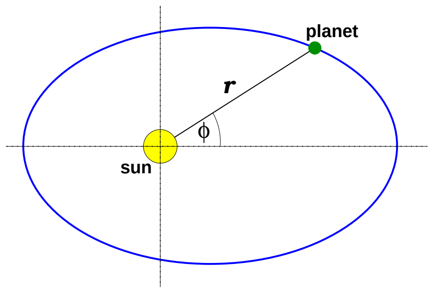
$$L(\phi, \dot{\phi}) = \frac{1}{2}ml^2\dot{\phi}^2 - mgl(1 - \cos\phi)$$

and the action integral is

$$F\{\phi\} = \int_{t_0}^{t_1} \left(\frac{1}{2}ml^2\dot{\phi}^2 - mgl(1 - \cos\phi) \right) dt$$

Kepler's problem of planetary motion

Single planet orbiting the sun.



Kinetic energy

$$\begin{aligned} T &= \frac{1}{2}m\left(\dot{x}^2(t) + \dot{y}^2(t)\right) \\ &= \frac{1}{2}m\left(\dot{r}^2(t) + r^2(t)\dot{\phi}^2(t)\right) \end{aligned}$$

Potential energy

$$V(r) = -\int f(r)\,dr = -\frac{GmM}{r(t)}$$

where the force $f = -\frac{dV}{dr} = -\frac{GmM}{r^2}$ (from Newton)

Hamilton's principle and EL eq.s

Hamilton's principle states we should look for curves along which the function

$$F\{\mathbf{q}\} = \int_{t_0}^{t_1} L(t, \mathbf{q}, \dot{\mathbf{q}})\,dt$$

is stationary. The Euler-Lagrange equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0$$

for all $k = 1, \ldots, n$, and so for mechanical systems, the Lagrangian satisfies these equations.

Newton's laws

Often the potential V depends only on location and time, and the kinetic energy depends only on the derivatives of the position, then the Euler-Lagrange equations reduce to

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} + \frac{\partial V}{\partial q_k} = 0$$

Given kinetic energy of the form $T(\dot{\mathbf{q}}) = \frac{1}{2}m\sum_i \dot{q}_i^2$, then the EL equations become

$$m\ddot{q}_k = -\frac{\partial V}{\partial q_k} = f_k = \text{ the force in direction } k$$

We have **derived** Newton's laws of motion, i.e. $\mathbf{f} = m\mathbf{a}$ from a more general principle.

Conservation laws

If the potential does not depend on time, the Lagrangian does not explicitly depend on t and so we may form $H(\mathbf{q}, \dot{\mathbf{q}})$ as before, i.e.

$$H(\mathbf{q}, \dot{\mathbf{q}}) = \sum_{k=1}^{n} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L = const$$

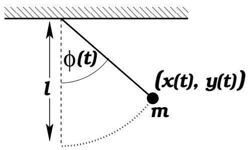
Given kinetic energy of the form $T(\dot{\mathbf{q}}) = \frac{1}{2} m \sum_i \dot{q}_i^2$, this becomes

$$H(\mathbf{q}, \dot{\mathbf{q}}) = 2T - L = T + V = const$$

Thus energy is conserved in such a system.

Example: a simple pendulum

$$F\{\phi\} = \int_{t_0}^{t_1} \left(\frac{1}{2} m l^2 \dot{\phi}^2 - mgl(1 - \cos\phi) \right) dt$$



The kinetic energy is in the appropriate form, and the potential does not depend on time, so the pendulum system conserves energy, e.g.

$$\frac{1}{2} m l^2 \dot{\phi}^2 + mgl(1 - \cos\phi) = const$$

Removing constant terms (where possible), we get

$$\dot{\phi}^2 - \frac{2g}{l} \cos\phi = c_1$$

Example: a simple pendulum

Given conservation of energy

$$\dot{\phi}^2 - \frac{2g}{l} \cos\phi = c_1$$

To solve, differentiate with respect to t

$$2\dot{\phi} \left[\ddot{\phi} + \frac{g}{l} \sin\phi \right] = 0$$

Assume that $\dot{\phi} \neq 0$, and multiply by m, and we get

$$m\ddot{\phi} + \frac{gm}{l} \sin\phi = 0$$

which is an equation relating torque to the rate of change of angular momentum

Example: a simple pendulum

$$\ddot{\phi} + \frac{g}{l} \sin\phi = 0$$

Motion is quite complicated. Small oscillations approximation $\sin\phi \simeq \phi$ we get

$$\ddot{\phi} + \frac{g}{l} \phi = 0$$

and so

$$\phi(t) = A \sin\left(\sqrt{\frac{g}{l}} t \right) + \phi_0$$

which has period $2\pi\sqrt{\frac{l}{g}}$

Brachystochrone in 3D

Find the curve of fastest descent between the points (x_0, y_0, z_0) and (x_1, y_1, z_1) where z is height, and x and y are spatial. Consider y and z to be functions of x. The time for the descent is

$$\sqrt{2g}T\{y,z\} = \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2+z'^2}}{\sqrt{z_0 - z}}\, dx$$

The Euler-Lagrange equations are

$$\begin{aligned} \frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2+z'^2}\sqrt{z_0-z}} \right) &= 0 \\ \frac{d}{dx}\left(\frac{z'}{\sqrt{1+y'^2+z'^2}\sqrt{z_0-z}} \right) - \frac{\sqrt{1+y'^2+z'^2}}{2(z_0-z)^{3/2}} &= 0 \end{aligned}$$

Brachystochrone in 3D

We can transform the first to get

$$\frac{y'}{\sqrt{1+y'^2+z'^2}} = c_1\sqrt{z_0 - z}$$

but the second EL equation is a mess. Instead, note that the function f is not explicitly dependent on x, and so we may derive a function $H(y, y', z, z') = const$ as before. In this case

$$-H(y,y',z,z') = f - y'\frac{\partial f}{\partial y'} - z'\frac{\partial f}{\partial z'} = c_2$$

Brachystochrone in 3D

$$\begin{aligned} -H(y,y',z,z') &= f - y'\frac{\partial f}{\partial y'} - z'\frac{\partial f}{\partial z'} \\ &= \frac{\sqrt{1+y'^2+z'^2}}{\sqrt{z_0-z}} - \frac{y'^2}{\sqrt{1+y'^2+z'^2}\sqrt{z_0-z}} - \frac{z'^2}{\sqrt{1+y'^2+z'^2}\sqrt{z_0-z}} \\ &= \frac{1+y'^2+z'^2-y'^2-z'^2}{\sqrt{1+y'^2+z'^2}\sqrt{z_0-z}} \\ &= \frac{1}{\sqrt{1+y'^2+z'^2}\sqrt{z_0-z}} = c_2 \end{aligned}$$

Brachystochrone in 3D

The two parts we have derived are

$$\begin{aligned} \frac{y'}{\sqrt{1+y'^2+z'^2}} &= c_1\sqrt{z_0-z} \\ \frac{1}{\sqrt{1+y'^2+z'^2}} &= c_2\sqrt{z_0-z} \end{aligned}$$

Divide the first, by the second, and we get

$$y' = \frac{c_1}{c_2} = const$$

from which we derive $y = \frac{c_1}{c_2}(x - x_1) + y_1$, which is the equation of a vertical plane. Thus the solutions in 3D can be reduced to the solution to the Brachystochrone in a 2D vertical plane (which is physically obvious).

Kepler's problem of planetary motion

Single planet orbiting the sun.

$$L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + \frac{GmM}{r}$$

Hamilton's principle says we have to find stationary curves of the integral of L, so we can jump straight to the E-L equations

$$\begin{aligned} \frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} &= 0 \\ \frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} &= 0 \end{aligned}$$

Kepler's problem of planetary motion

E-L equations $L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + \frac{GmM}{r}$

$$\begin{aligned} \frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} &= 0 \\ \frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} &= 0 \end{aligned}$$

give

$$\begin{aligned} mr\dot{\phi}^2 - \frac{GmM}{r^2} - m\frac{d}{dt}\dot{r} &= 0 \\ m\frac{d}{dt}r^2\dot{\phi} &= 0 \end{aligned}$$

Equations of planetary motion

Simplify (assuming $m \neq 0$ and $r \neq 0$)

$$\begin{aligned} mr\dot{\phi}^2 - \frac{GmM}{r^2} - m\frac{d}{dt}\dot{r} &= 0 \\ m\frac{d}{dt}r^2\dot{\phi} &= 0 \end{aligned}$$

to get

$$\begin{aligned} \ddot{r} - r\dot{\phi}^2 &= -\frac{GM}{r^2} \\ \dot{\phi}r^2 &= c \end{aligned}$$

Interesting aside

The equation $\dot{\phi}r^2 = c$, gives the angular velocity $\dot{\phi}$ in terms of distance from the sun, but also allows us to determine the velocity at right angles to the direction of the sun as

$$v_r = r\dot{\phi} = c/r$$

So we can calculate the angular momentum

$$p_a = rm\dot{\phi} = cm$$

which is constant (as you might expect).
The law also allows one to derive Kepler's second law (the arc of an orbit over equal periods of time traverse equal areas).

Solving the equations

First equation, including the condition $\dot{\phi} = c/r^2$ gives

$$\begin{aligned} \ddot{r} - r\dot{\phi}^2 &= -\frac{GM}{r^2} \\ \ddot{r} - \frac{c^2}{r^3} &= -\frac{GM}{r^2} \end{aligned}$$

Now instead of calculating this in terms of derivatives with respect to time, lets convert to derivatives with respect to ϕ. Denote such derivatives using, e.g., r'

$$\dot{r} = \frac{dr}{d\phi}\frac{d\phi}{dt} = r'\dot{\phi}$$

Solving the equations

From the chain rule and $\dot{\phi} = c/r^2$ we get

$$\begin{aligned} \dot{r} &= \frac{dr}{d\phi}\frac{d\phi}{dt} = r'\dot{\phi} \\ \ddot{r} &= \frac{d}{d\phi}\left(r'\dot{\phi}\right)\frac{d\phi}{dt} \\ &= \frac{d}{d\phi}\left(\frac{cr'}{r^2}\right)\dot{\phi} \\ &= \left[\frac{cr''}{r^2} - \frac{2cr'^2}{r^3}\right]\dot{\phi} \\ &= \frac{c^2}{r^2}\left[\frac{r''}{r^2} - \frac{2r'^2}{r^3}\right] \end{aligned}$$

Solving the equations

Substitute the above form of $\ddot{r}$ into the first DE and we get

$$\begin{aligned} \ddot{r} - \frac{c^2}{r^3} &= -\frac{GM}{r^2} \\ \frac{c^2}{r^2}\left[\frac{r''}{r^2} - \frac{2r'^2}{r^3}\right] - \frac{c^2}{r^3} &= -\frac{GM}{r^2} \end{aligned}$$

Once again note that $r \neq 0$, and $\dot{\phi} \neq 0$ for all but degenerate orbits (straight lines through the origin), so that we can multiply by r^2/c^2 to get

$$\frac{r''}{r^2} - \frac{2r'^2}{r^3} - \frac{1}{r} = -\frac{GM}{c^2}$$

Solving the equations

Take the substitution $u = p/r$ and then

$$\begin{aligned} u' &= -\frac{pr'}{r^2} \\ u'' &= -\frac{pr''}{r^2} + \frac{2pr'^2}{r^3} \end{aligned}$$

Now note that in our equation for r' we get

$$\begin{aligned} \frac{r''}{r^2} - \frac{2r'^2}{r^3} - \frac{1}{r} &= -\frac{GM}{c^2} \\ -\frac{u''}{p} - \frac{u}{p} &= -\frac{GM}{c^2} \\ u'' + u &= \frac{GMp}{c^2} \end{aligned}$$

Solving the equations

The equation

$$u'' + u = k$$

has a simple solution. The homogeneous form has the solution

$$u = A\cos(\phi - \omega)$$

for some constants A and ω and the particular solution is

$$u = k$$

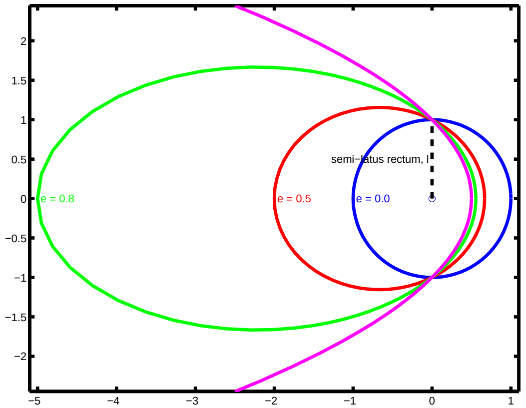
So the final solution can be scaled to give

$$\frac{L}{r} = 1 + e\cos(\phi - \omega)$$

This is just the equation of a conic section.

Possible trajectories

- $e = 0$: circle
- $0 < e < 1$: ellipse
- $e = 1$: parabola
- $e > 1$: hyperbola



L is the semi-latus rectum (dashed line), e is the eccentricity, and ω gives the angle of the perihelion (point of closest approach) which is zero in the above figure.