Variational Methods & Optimal Control

lecture 25

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Conservation Laws

One of the more exciting things we can derive relates to fundamental physics laws: conservation of energy, momentum, and angular momentum. We can now derive all of these from an underlying principle: Noether's theorem.

Hamilton’s principle

We now have a group of equivalent methods

- Euler-Lagrange equations
- Hamilton’s equations
- Hamilton-Jacobi equation

We saw earlier that these can give us other methods

- Hamilton’s principle $\Rightarrow$ Newton’s laws of motion
- When L is not explicitly dependent on t, then the Hamiltonian H is constant in time.
  - conservation of energy
  - this is an illustration of a symmetry in the problem appearing in the Hamiltonian

Conservation laws

Given the functional

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y', \ldots, y^{(n)})\, dx$$

if there is a function $\phi(x, y, y', \ldots, y^{(k)})$ such that

$$\frac{d}{dx}\phi(x, y, y', \ldots, y^{(k)}) = 0$$

for all extremals of F, then this is called a ***k*th order conservation law**

- use obvious extension for functionals of several dependent variables.

Conservation law example

Given the functional

$$F\{y\} = \int_{x_0}^{x_1} f(y, y')\, dx$$

where f is not explicitly dependent on t, we know that the Hamiltonian

$$H = y' \frac{\partial f}{\partial y'} - f$$

is constant, and so

$$\frac{dH}{dx} = 0$$

is a first order conservation law for the system.

Several independent variables

For functionals of several independent variables, e.g.

$$F\{z\} = \iint_{\Omega} z(x,y)\, dx\, dy$$

the equivalent conservation law is

$$\nabla \cdot \phi = 0$$

For some function $\phi(x, y, z, z', \ldots, z^{(k)})$.

- Results here can be extended to these cases, but we won't look at them here.

Conservation laws

- physically interesting
  - tell you about system of interest
- can simplify solution
  - $\phi(x, y, y', \ldots, y^{(k)}) = const$ is an order k DE, rather than E-L equations which are order $2n$
- $\phi(x, y, y', \ldots, y^{(k)}) = const$ is often called the **first integral** of the E-L equations
  - RHS is a constant of integration (determined by boundary conditions)
- how do we find conservation laws?
  - Noether's theorem

Variational symmetries

The key to finding conservation laws lies in finding symmetries in the problem.

- "symmetries" are the result of transformations under which the functional is invariant
- E.G. time invariance symmetry results in constant H
- more generally, take a parameterized family of smooth transforms

$$X = \theta(x, y; \varepsilon), \qquad Y = \phi(x, y; \varepsilon)$$

  where

$$x = \theta(x, y; 0), \qquad y = \phi(x, y; 0)$$

  e.g. we get the identity transform for $\varepsilon = 0$

- examples **translations** and **rotations**

Jacobian

The Jacobian is

$$J = \begin{vmatrix} \theta_x & \theta_y \\ \phi_x & \phi_y \end{vmatrix} = \theta_x\phi_y - \theta_y\phi_x$$

- **smooth:** if functions x and y have continuous partial derivatives.
- **non-singular:** if Jacobian is non-zero (and hence an inverse transform exists)

Now for $\varepsilon = 0$, we require the identity transform, so $J = 1$. Also, we require a smooth transform, so J is a smooth function of ε, and so for sufficiently small $|\varepsilon|$, the transform is non-singular.

Example transformations

- **translations** (ε is the translation distance)

$$\begin{array}{lrclcrcl} & X & = & x+\varepsilon & \quad & Y & = & y \\ \text{or} & X & = & x & & Y & = & y+\varepsilon \end{array}$$

both have Jacobian

$$J = 1$$

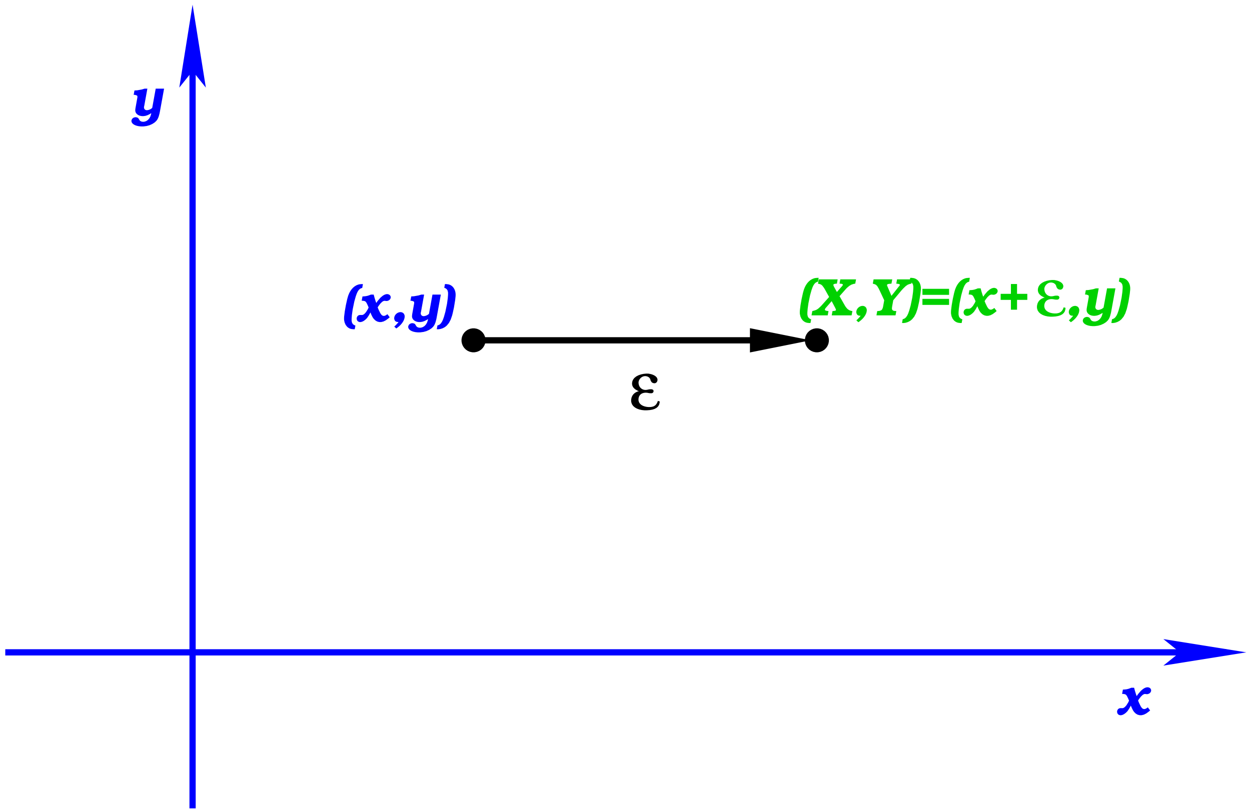
and inverse transformations

$$\begin{array}{lrclcrcl} & x & = & X-\varepsilon & \quad & y & = & Y \\ \text{or} & x & = & X & & y & = & Y-\varepsilon \end{array}$$

Example transformations

- **translations** (ε is the translation distance)

$$X = x+\varepsilon \qquad Y = y$$



Example transformations

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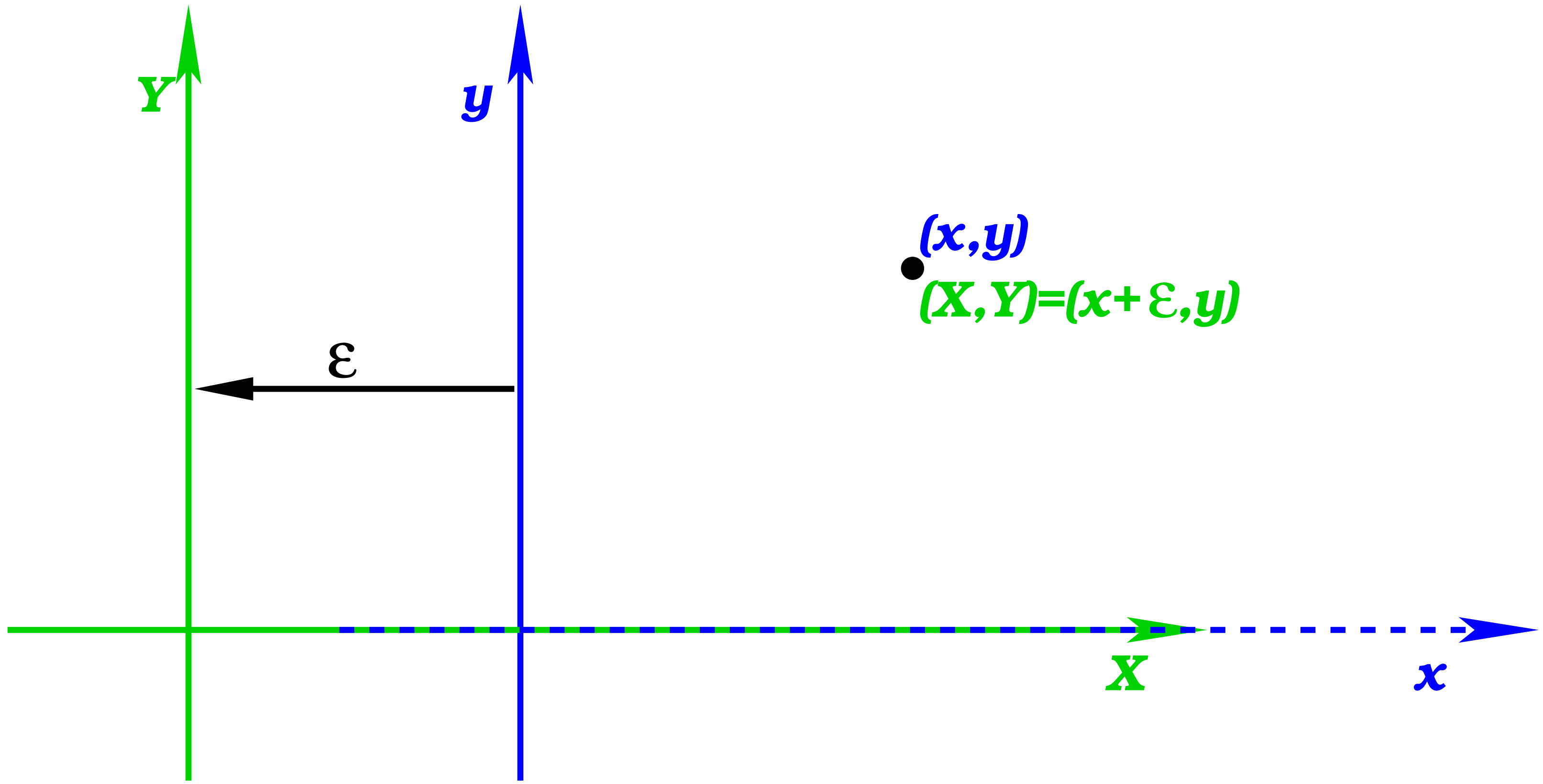


Example transformations

- **rotations** (ε is the rotation angle)

$$X = x\cos\varepsilon + y\sin\varepsilon \qquad Y = -x\sin\varepsilon + y\cos\varepsilon$$

has Jacobian

$$J = \cos^2\varepsilon + \sin^2\varepsilon = 1$$

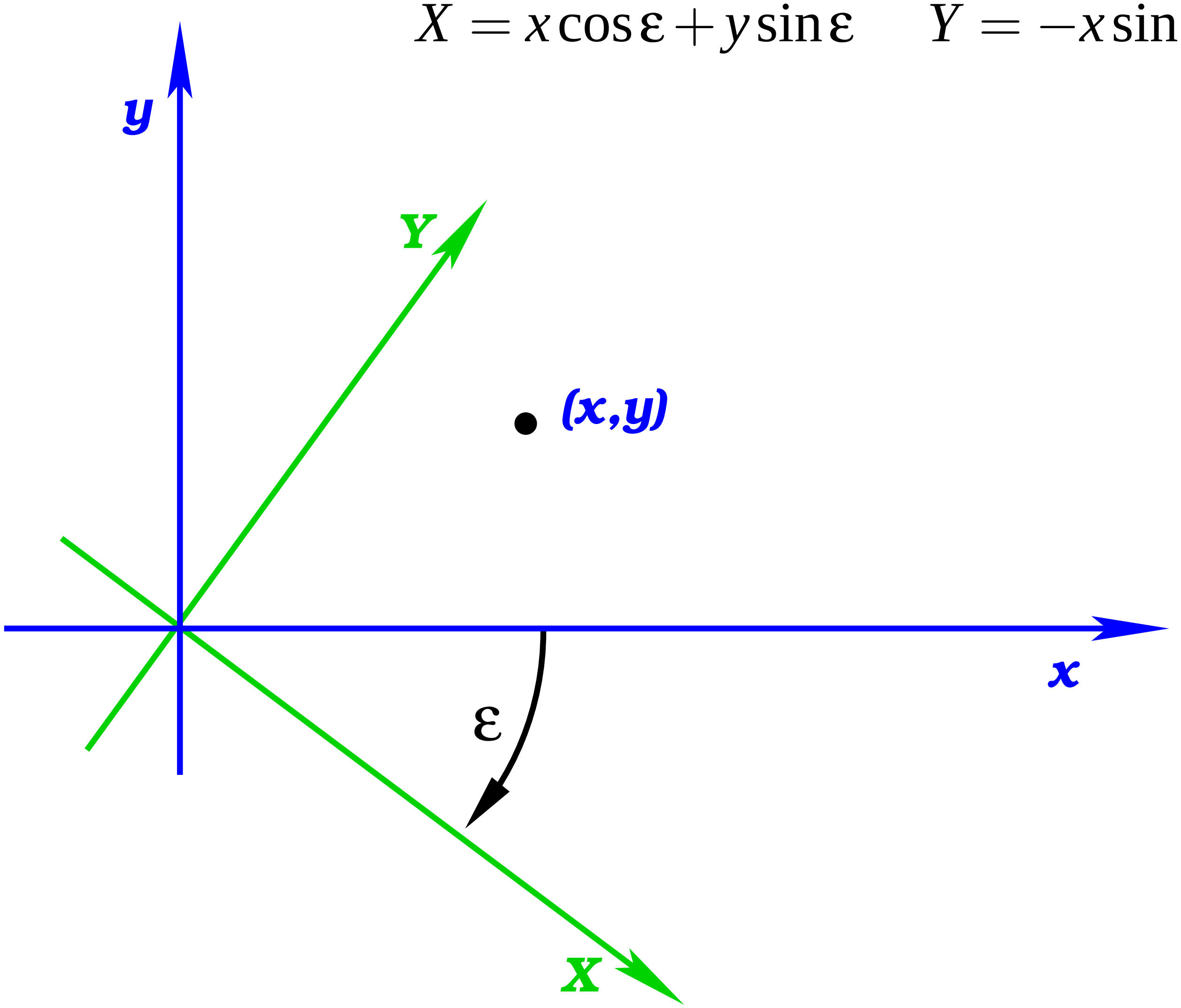
and inverse

$$x = X\cos\varepsilon - Y\sin\varepsilon \qquad y = X\sin\varepsilon + Y\cos\varepsilon$$

Example transformations

- **rotations** (ε is the rotation angle)

$$X = x\cos\varepsilon + y\sin\varepsilon \qquad Y = -x\sin\varepsilon + y\cos\varepsilon$$



Example transformations

- **rotations** (ε is the rotation angle)

$$X = x\cos\varepsilon + y\sin\varepsilon \qquad Y = -x\sin\varepsilon + y\cos\varepsilon$$

To derive this, change coordinates to polar coordinates

$$x = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta)$$

Under a rotation by ε, the new coordinates (X,Y) are

$$X = r\cos(\theta-\varepsilon) \quad \text{and} \quad Y = r\sin(\theta-\varepsilon)$$

Use trig. identities $\cos(u-v) = \cos u\cos v + \sin u\sin v$ and $\sin(u-v) = \sin u\cos v - \cos u\sin v$, to get

$$\begin{aligned} X &= r\cos(\theta)\cos(\varepsilon) + r\sin(\theta)\sin(\varepsilon) = x\cos(\varepsilon) + y\sin(\varepsilon) \\ Y &= r\sin(\theta)\cos(\varepsilon) - r\cos(\theta)\sin(\varepsilon) = y\cos(\varepsilon) - x\sin(\varepsilon) \end{aligned}$$

Transformation of a function

Given a function $y(x)$, we can rewrite $Y(X)$ using the inverse transformation, e.g.

$$\phi^{-1}(X, Y(X); \varepsilon) = y(x) = y(\theta^{-1}(X, Y; \varepsilon))$$

For example, taking the curve $y = x$ under rotations

$$X \sin\varepsilon + Y \cos\varepsilon = X \cos\varepsilon - Y \sin\varepsilon$$

which we rearrange to get

$$Y(X) = \frac{\cos\varepsilon - \sin\varepsilon}{\cos\varepsilon + \sin\varepsilon} X$$

Similarly we can derive $Y'(X)$

Transform invariance

If

$$\int_{x_0}^{x_1} f(x, y, y'(x))\, dx = \int_{X_0}^{X_1} f(X, Y, Y'(X))\, dX$$

for all smooth functions $y(x)$ on $[x_0, x_1]$ then we say that the functional in invariant under the transformation.

- also called **variational invariance**
- The transform is called a **variational symmetry**
- Related to conservation laws

Also note that the E-L equations are invariant under such a transform, e.g. they produce the same extremal curves.

Infinitesimal generators

For small ε we can use Taylor's theorem to write

$$\begin{aligned} X &= \theta(x,y;0) + \varepsilon \left.\frac{\partial\theta}{\partial\varepsilon}\right|_{(x,y;0)} + O(\varepsilon^2) \\ Y &= \phi(x,y;0) + \varepsilon \left.\frac{\partial\phi}{\partial\varepsilon}\right|_{(x,y;0)} + O(\varepsilon^2) \end{aligned}$$

Define the infinitesimal generators

$$\xi(x,y) = \left.\frac{\partial\theta}{\partial\varepsilon}\right|_{(x,y;0)} \qquad \eta(x,y) = \left.\frac{\partial\phi}{\partial\varepsilon}\right|_{(x,y;0)}$$

and then for small ε

$$\begin{aligned} X &\simeq x + \varepsilon\xi \\ Y &\simeq y + \varepsilon\eta \end{aligned}$$

Examples

- **translations:**

$$
\begin{array}{rrcccccc}
 & (X,Y) & = & (x+\varepsilon,y) & \Rightarrow & (\xi,\eta) & = & (1,0) \\
\text{or} & (X,Y) & = & (x,y+\varepsilon) & \Rightarrow & (\xi,\eta) & = & (0,1)
\end{array}
$$

- **rotations:**

$$X = \theta(x,y;\varepsilon) = x\cos\varepsilon + y\sin\varepsilon \qquad Y = \phi(x,y;\varepsilon) = -x\sin\varepsilon + y\cos\varepsilon$$

So

$$
\begin{array}{rcccccc}
\xi & = & \left.\dfrac{\partial\theta}{\partial\varepsilon}\right|_{\varepsilon=0} & = & -x\sin\varepsilon + y\cos\varepsilon\big|_{\varepsilon=0} & = & y \\
\eta & = & \left.\dfrac{\partial\phi}{\partial\varepsilon}\right|_{\varepsilon=0} & = & -x\cos\varepsilon - y\sin\varepsilon\big|_{\varepsilon=0} & = & -x
\end{array}
$$

Emmy Noether

- Amalie Emmy Noether, 23 March 1882 – 14 April 1935
- Described by Einstein and many others as the most important woman in the history of mathematics.
- Most of her work was in algebra
- Worked at the Mathematical Institute of Erlangen without pay for seven years
- Invited by David Hilbert and Felix Klein to join the mathematics department at the University of Göttingen, a world-renowned center of mathematical research. The philosophicqal faculty objected, however, and she spent four years lecturing under Hilbert's name.

Noether's theorem

Suppose the $f(x,y,y')$ is variationally invariant on $[x_0, x_1]$ under a transform with infinitesimal generators ξ and η, then

$$\eta p - \xi H = const$$

along any extremal of

$$F\{y\} = \int_{x_0}^{x_1} f(x,y,y')\, dx$$

Example (i)

Invariance in translations in x, i.e.

$$\begin{aligned} (X, Y) &= (x + \varepsilon, y) \\ (\xi, \eta) &= (1, 0) \end{aligned}$$

So, a system with such invariance has

$$H = const$$

which is what we showed earlier regarding functionals with no explicit dependence on x.

Example (ii)

Invariance in translations in y, i.e.

$$\begin{aligned}(X,Y) &= (x, y+\varepsilon)\\ (\xi,\eta) &= (0,1)\end{aligned}$$

So, a system with such invariance has

$$p = const$$

which is what we showed earlier regarding functionals with no explicit dependence on y.

More than one dependent variable

Transforms with more than one dependent variable

$$\begin{aligned} T &= \theta(t, \mathbf{q}; \varepsilon) \\ Q_k &= \phi_k(t, \mathbf{q}; \varepsilon) \end{aligned}$$

and the infinitesimal generators are

$$\begin{aligned} \xi &= \left. \frac{\partial \theta}{\partial \varepsilon} \right|_{\varepsilon=0} \\ \eta_k &= \left. \frac{\partial \phi_k}{\partial \varepsilon} \right|_{\varepsilon=0} \end{aligned}$$

More than one dependent variable

Noether's theorem: Suppose $L(t, \mathbf{q}, \dot{\mathbf{q}})$ is variationally invariant on $[t_0, t_1]$ under a transform with infinitesimal generators ξ and η_k. Given

$$p = \frac{\partial L}{\partial \dot{q}_k}, \qquad H = \sum_{k=1}^{n} p_k \dot{q}_k - L$$

Then

$$\sum_{k=1}^{n} p_k \eta_k - H\xi = const$$

along any extremal of

$$F\{\mathbf{q}\} = \int_{t_0}^{t_1} L(t, \mathbf{q}, \dot{\mathbf{q}})\, dt$$

Example: rotations

Invariance in rotations, i.e.

$$\begin{aligned}(T, Q_1, Q_2) &= (t, q_1 \cos \varepsilon + q_2 \sin \varepsilon, -q_1 \sin \varepsilon + q_2 \cos \varepsilon) \\ (t, q_1, q_2) &= (T, Q_1 \cos \varepsilon - Q_2 \sin \varepsilon, Q_1 \sin \varepsilon + Q_2 \cos \varepsilon)\end{aligned}$$

The infinitesimal generators are

$$\begin{aligned}\xi &= 0 \\ \eta_1 &= -q_1 \sin \varepsilon + q_2 \cos \varepsilon \big|_{\varepsilon=0} = q_2 \\ \eta_2 &= -q_1 \cos \varepsilon - q_2 \sin \varepsilon \big|_{\varepsilon=0} = -q_1\end{aligned}$$

So, a system with such invariance has

$$\sum_{i=1}^{2} p_i \eta_i - H\xi = p_1 q_2 - p_2 q_1 = const$$

So **angular momentum** in conserved.

Common symmetries

Given a system in 3D with Kinetic Energy $T(\dot{\mathbf{q}}) = \frac{1}{2}m\left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2\right)$, and Potential Energy $V(t, \mathbf{q})$.

- invariance of L under time translations corresponds to conservation of Energy
- invariance of L under spatial translations corresponds to conservation of momentum
- invariance of L under rotations corresponds to conservation of angular momentum

Finding symmetries

Testing for non-trivial symmetries can be tricky.
Useful result is the *Rund-Trautman identity*:
It leads also to a simple proof of Noether's theorem

More advanced cases

- Laplace-Runge-Lenz vector in planetary motion corresponds to rotations of 3D sphere in 4D
- symmetries in general relativity
- symmetries in quantum mechanics
- symmetries in fields