Variational Methods & Optimal Control

lecture 27

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Bang-Bang controllers and other related issues

Here we consider more generally what conditions result in a bang-bang controller.

Bang-Bang controllers

A linear optimal control problem is one in which the **control variables u** enter the Hamiltonian linearly, e.g.

$$H = \psi(\mathbf{x}, \mathbf{p}, t) + \sigma(\mathbf{x}, \mathbf{p}, t)^T \mathbf{u}(t)$$

Examples:

- a time minimization problem, with linear state equation
$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$
- the optimal economic growth model with $U(c) = c$, so the functional is
$$F\{c\} = \int_0^T c(t) e^{-rt}\, dt$$
subject to $\dot{k} = f(k) - \lambda k - c$ leads to the Hamiltonian
$$H = \left(e^{-rt} - p\right) c + p\left(f(k) - \lambda k\right)$$

Bang-Bang controllers

In general (for a linear problem) there will be no extremal unless the control is bounded, e.g. $m_i \leq u_i \leq M_i$, but where m_i and M_i are constant, we can re-scale the problem to consider bounded controls $|\tilde{u}_i| \leq 1$, by taking

$$\tilde{u}_i = 2\frac{u_i - m_i}{M_i - m_i} - 1$$

When the PMP is applied to this type of problem the optimal control is

$$u_i(t) = \begin{cases} 1, & \text{if } \sigma_i > 0 \\ -1 & \text{if } \sigma_i < 0 \end{cases}$$

Where $\sigma_i \neq 0$ is a **bang-bang** controller (otherwise it is singular), and σ_i is a **switching function**

Explanation

Consider a linear problem with one control u, then

$$H = \psi(\mathbf{x}, \mathbf{p}, t) + \sigma(\mathbf{x}, \mathbf{p}, t)u$$

- The PMP requires us to maximize H for all u.
- The derivative of H WRT to u is $\sigma(\mathbf{x}, \mathbf{p}, t)$.
- If $\sigma(\mathbf{x}, \mathbf{p}, t) \neq 0$ the derivative is never zero.
- Hence the maximum will occur at the bounds of u.
- If $\sigma(\mathbf{x}, \mathbf{p}, t) > 0$, the maximum will occur for the positive bound of u, whereas if $\sigma(\mathbf{x}, \mathbf{p}, t) < 0$ the maximum will occur for the negative bound.
- Hence σ is a switching function.

Example: optimal fish harvesting

- fish stock (population $x(t)$)
- grows at a fixed rate γ, so without harvesting

$$\dot{x} = \gamma x$$

- harvesting at rate u reduces the population

$$\dot{x} = \gamma x - u$$

- we wish to harvest the maximum number of fish in time T,
  - discount by rate r for future harvests
  - maximize

$$F\{u\} = \int_0^T u e^{-rt}\, dt$$

Example: optimal fish harvesting

Problem formulation: Maximize

$$F\{u\} = \int_0^T u e^{-rt}\, dt$$

subject to

$$\dot{x} = \gamma x - u$$

and $x(0) = 1$, and $x(T)$ free.

Equivalent problem: Minimize

$$F\{u\} = \int_0^T -u e^{-rt}\, dt$$

subject to

$$\dot{x} = \gamma x - u$$

and $x(0) = 1$, and $x(T)$ free.

Example: optimal fish harvesting

The Hamiltonian is

$$H = u e^{-rt} + p(\gamma x - u)$$

which is linear in the control variable.
Hamilton's equations (the canonical, or co-state equations) are

$$\frac{\partial H}{\partial p} = \frac{dx}{dt} \quad \text{and} \quad \frac{\partial H}{\partial x} = -\frac{dp}{dt}$$

The first of Hamilton's equations just gives back the growth equation $\dot{x} = \gamma x - u$, the second gives

$$\frac{\partial H}{\partial x} = \gamma p = -\frac{dp}{dt}$$

which has solution $p = c_1 e^{-\gamma t}$.

Example: optimal fish harvesting

The Hamiltonian is

$$\begin{aligned} H &= ue^{-rt} + p(\gamma x - u) \\ &= p\gamma x + \left[e^{-rt} - p\right] u \end{aligned}$$

which is linear in the control variable. The control must be bounded, and will be bang-bang with switching function

$$\sigma = e^{-rt} - p = e^{-rt} - c_1 e^{-\gamma t}$$

For $0 \leq u \leq 1$ we get $u = 0$ or 1.

$$u(t) = \begin{cases} 1, & \text{if } \sigma_i > 0 \\ 0, & \text{if } \sigma_i < 0 \end{cases}$$

Example: optimal fish harvesting

Given fixed end-time T, but free $x(T)$, then the natural boundary condition is $p(T) = 0$, so $c_1 = 0$, and

$$\sigma = e^{-rt} - c_1 e^{-\gamma t} = e^{-rt} > 0$$

- result is fishing at maximum rate
- if the fishing rate u is greater than the growth rate γx then the fish stock will eventually die out.

This model may be a big simplification (ignores economic factors like cost of fishing, or demand), but it does show some interesting features.

- **control is needed, or you get over-fishing!**

Time Minimization Problem

Time minimization, the functional to minimize is

$$T\{\mathbf{x}, \mathbf{u}\} = \int_{t_0}^{t_1} 1\, dt$$

Given that the starting state is $\mathbf{x}(t_0) = \mathbf{x}_0$, and the desired end state is $\mathbf{x}(t_1) = \mathbf{x}_1$, but that t_1 is not fixed, and $\mathbf{x}$ is subject to some DE

$$\dot{\mathbf{x}} = g(\mathbf{x}, \mathbf{u}, t)$$

To get a linear autonomous problem, we need that

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

Time Minimization Problem

Linear autonomous time minimization, the functional to minimize is

$$T\{\mathbf{x}, \mathbf{u}\} = \int_{t_0}^{t_1} 1\, dt$$

subject to

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

where A is a $n \times n$ constant matrix, and B is a $n \times m$ constant matrix. The controller is assumed to be bounded, e.g.

$$|u_i| \leq 1, \quad \text{for } i = 1, \ldots, m$$

The Hamiltonian and generalized momentum will be

$$H = -1 + \mathbf{p}^T A\mathbf{x} + \mathbf{p}^T B\mathbf{u} \quad \text{and} \quad \dot{\mathbf{p}} = -H_{\mathbf{x}} = -A^T \mathbf{p}$$

which is linear in the controller $\mathbf{u}$.

Time Minimization Problem

We know the control will governed by the **switching function**

$$\sigma = \mathbf{p}^T B$$

so we get the control

$$u_i(t) = \begin{cases} 1, & \text{if } \mathbf{p}^T \mathbf{b}_i > 0 \\ -1 & \text{if } \mathbf{p}^T \mathbf{b}_i < 0 \\ \text{unknown} & \text{if } \mathbf{p}^T \mathbf{b}_i = 0 \end{cases}$$

where the $\mathbf{b}_i$ are the m columns of the matrix B. Given $\dot{\mathbf{p}} = -A^T \mathbf{p}$, so $\mathbf{p} = e^{-A^T(t-t_0)}\mathbf{p}_0$, it is unlikely that $\mathbf{p}^T \mathbf{b}_i = 0$, so singular control is ruled out, and the control is bang-bang.

Time Minimization Problem

In general a control may or may not exist!

- **existence:** If A is a stable matrix (i.e., all the eigenvalues of A have non-positive real parts), then for any point $\mathbf{x}_0$, there exists an optimal control which will go from $\mathbf{x}_0$ to the origin.

  This is useful because we can rewrite the problem so that the desired end-point $\mathbf{x}(t_1) = \mathbf{0}$.
- **uniqueness:** If an optimal control exists, it is unique.
- **switching:** If the eigenvalues of the $n \times n$ matrix A are all real, then there exists a unique control control, where each $u_i = \pm 1$ is piecewise constant and has no more than $n-1$ switchings.

Time Minimization Problem

Example: parking problem (from Lecture 19)

Rewrite the problem so the point B is at the origin ($x(t_1) = 0$), and the control $u =$ Force/mass is bounded by $|u| \leq 1$. The differential equation

$$\ddot{x} = u$$

can be written as two first order DEs by defining $x_1 = x$ and $x_2 = \dot{x}$, so that

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

Time Minimization Problem

The matrix A has eigenvalues $\lambda = 0, 0$, and so satisfies the existence and uniqueness conditions. The Hamiltonian is

$$H = -1 + p_1 x_2 + p_2 u$$

So the switching function is p_2. Hamilton's equations (PMP) results in

$$\begin{aligned} \dot{p}_1 &= -\frac{\partial H}{\partial x_1} = 0 \\ \dot{p}_2 &= -\frac{\partial H}{\partial x_2} = -p_1 \end{aligned}$$

with solution (c_1 and c_2 are constants of integration)

$$\begin{aligned} p_1 &= c_1 \\ p_2 &= -c_1 t + c_2 \end{aligned}$$

Time Minimization Problem

The switching function $p_2 = -c_1 t + c_2$ is guaranteed to change sign at most $n - 1 = 1$ times, so the possible solutions are

$$
\begin{aligned}
u &= 1 \text{ for all } t \in [0, T] \\
u &= -1 \text{ for all } t \in [0, T] \\
u &= \begin{cases} -1 \text{ for all } t \in [0, t_s) \\ 1 \text{ for all } t \in (t_s, T] \end{cases} \\
u &= \begin{cases} 1 \text{ for all } t \in [0, t_s) \\ -1 \text{ for all } t \in (t_s, T] \end{cases}
\end{aligned}
$$

Time Minimization Problem

Solving the DE for $u = \pm 1$

$$
\begin{aligned}
x_2 &= \pm t + c_3 \\
x_1 &= \pm \frac{1}{2} t^2 + c_3 t + c_4
\end{aligned}
$$

Time can be eliminated from the above by squaring the first equation and multiplying by 1/2,

$$
\begin{aligned}
\frac{1}{2} x_2^2 &= \frac{1}{2} t^2 \pm c_3 t + \frac{1}{2} c_3^2 \\
x_1 &= \pm \frac{1}{2} t^2 + c_3 t + c_4
\end{aligned}
$$

Time Minimization Problem

For $u = \pm 1$

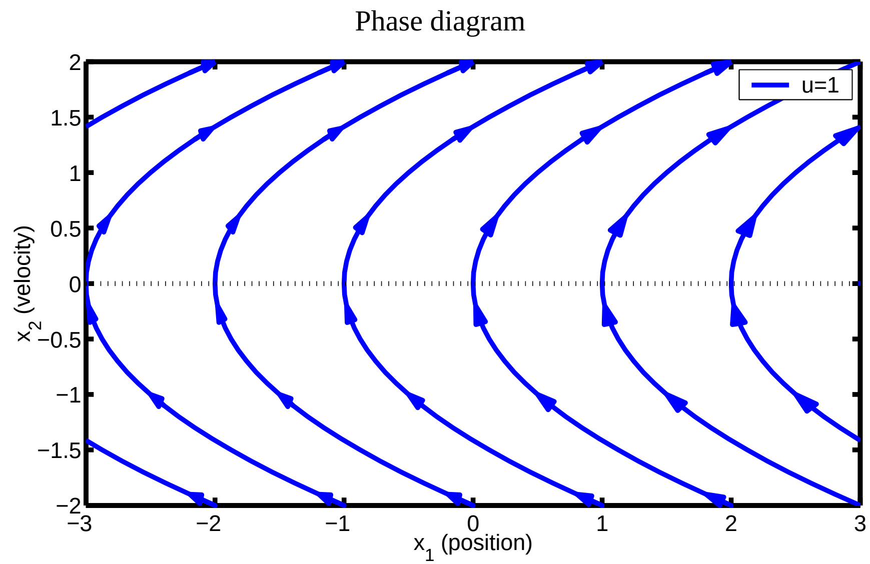
$$
\begin{aligned}
\frac{1}{2} x_2^2 &= \frac{1}{2} t^2 \pm c_3 t + \frac{1}{2} c_3^2 \\
x_1 &= \pm \frac{1}{2} t^2 + c_3 t + c_4
\end{aligned}
$$

so we can write x_1 as a function of x_2

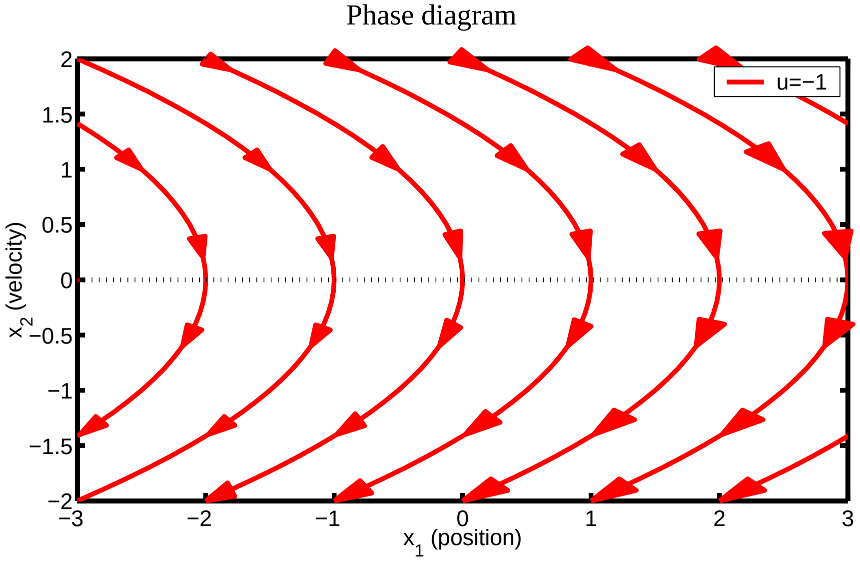
$$
x_1 = \begin{cases} \frac{1}{2} x_2^2 + c_5 & \text{for } u = 1 \\ -\frac{1}{2} x_2^2 + c_6 & \text{for } u = -1 \end{cases}
$$

where $c_5 = c_4 - \frac{1}{2} c_3^2$ and $c_6 = c_4 + \frac{1}{2} c_3^2$

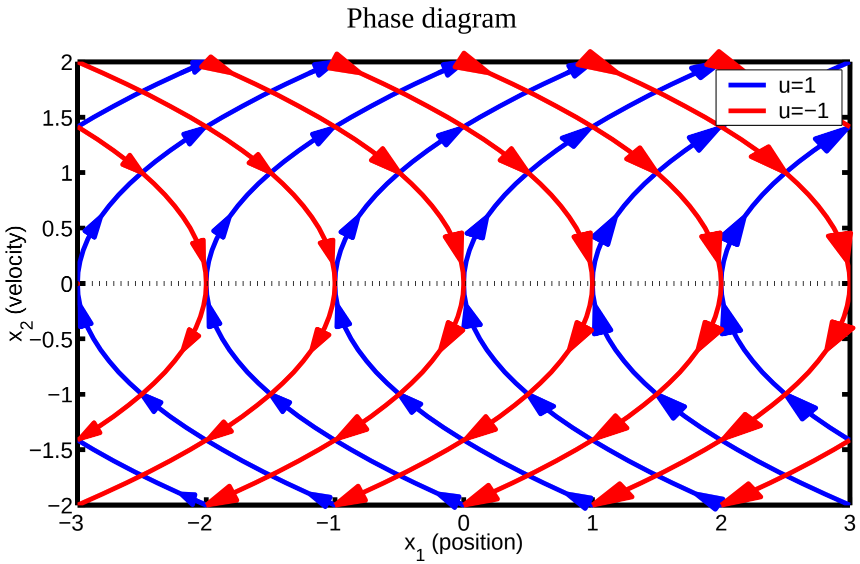
Phase diagram 1



Phase diagram 2



Combined phase diagram



Time Minimization Problem

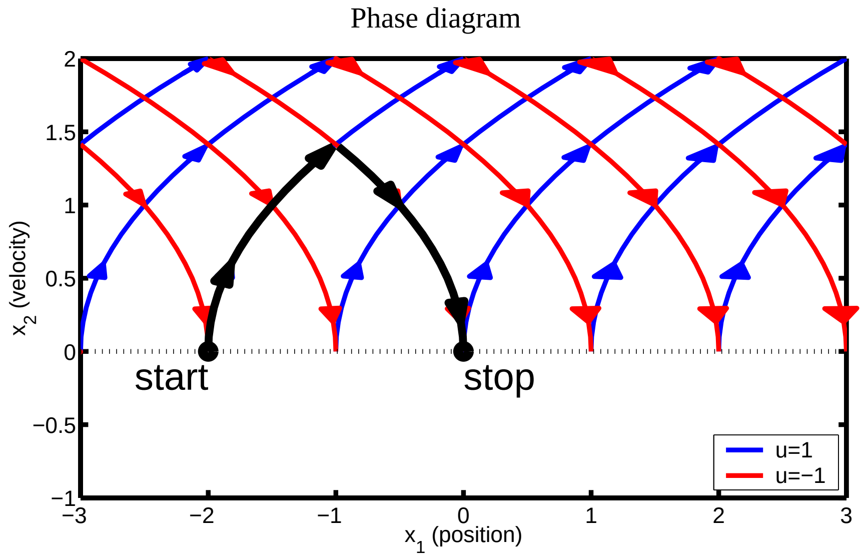
Parking problem: moving from point A (at $x = -2$) to B (at $x = 0$) and be stationary at both start and stop times. Given

$$x_1 = \text{position}$$
$$x_2 = \text{velocity}$$

the end-point conditions are

$$x_1(0) = -2 \qquad x_1(T) = 0$$
$$x_2(0) = 0 \qquad x_2(T) = 0$$

Time Minimization Problem



Time Minimization Problem

So the solution is case (4)

$$u = \begin{cases} 1 \text{ for all } t \in [0, t_s) \\ -1 \text{ for all } t \in (t_s, T] \end{cases}$$

Hence we know that the initial trajectory will be

$$\begin{aligned} x_2 &= t + c_3 \\ x_1 &= \frac{1}{2}t^2 + c_3 t + c_4 \end{aligned}$$

with $x_1(0) = -2$ and $x_2(0) = 0$, so $c_3 = 0$ and $c_4 = -2$, with result (for $t < t_s$)

$$\begin{aligned} x_2 &= t \\ x_1 &= \frac{1}{2}t^2 - 2 \end{aligned}$$

Time Minimization Problem

So the solution is case (4)

$$u = \begin{cases} 1 \text{ for all } t \in [0, t_s) \\ -1 \text{ for all } t \in (t_s, T] \end{cases}$$

Hence we know that the final trajectory will be

$$\begin{aligned} x_2 &= -t + c_3' \\ x_1 &= -\frac{1}{2}t^2 + c_3' t + c_4' \end{aligned}$$

with $x_1(T) = 0$ and $x_2(T) = 0$, so $c_3' = T$ and $c_4' = -T^2/2$, with result that for $t_s < t \leq T$

$$\begin{aligned} x_2 &= T - t \\ x_1 &= -\frac{(T-t)^2}{2} \end{aligned}$$

Time Minimization Problem

At time t_s the two paths must join, so we get the conditions

$$\begin{aligned} \lim_{t \to t_s^-} x_1(t) &= \lim_{t \to t_s^+} x_1(t) \\ \lim_{t \to t_s^-} x_2(t) &= \lim_{t \to t_s^+} x_2(t) \end{aligned}$$

When we substitute the initial and final paths, we get

$$\begin{aligned} \frac{1}{2}t_s^2 - 2 &= -\frac{(T-t_s)^2}{2} \\ t_s &= T - t_s \end{aligned}$$

The second equation requires that $t_s = T/2$, which we can observe directly from the symmetry of the phase diagram.

Time Minimization Problem

The continuity conditions are

$$\begin{aligned} \frac{1}{2}t_s^2 - 2 &= \frac{(T-t_s)^2}{2} \\ t_s &= T - t_s \end{aligned}$$

Given $t_s = T/2$ the first equation becomes

$$\frac{1}{8}T^2 - 2 = -\frac{T^2}{8}$$

which we rearrange to get

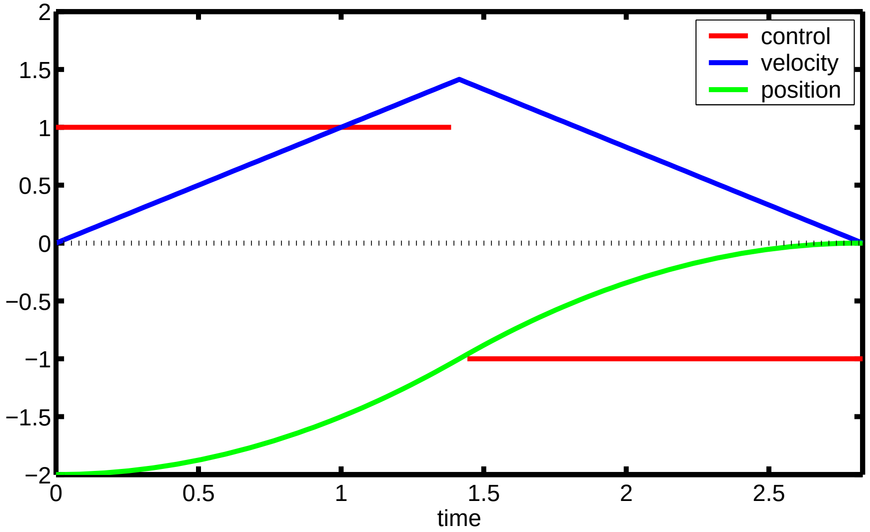
$$T^2 = 8$$

From the problem formulation $T > 0$, and so we take

$$T = 2\sqrt{2} \quad \text{and} \quad t_s = \sqrt{2}$$

Time Minimization Problem

Solution relative to time



Singular control

Linear problem,

$$H = \psi(\mathbf{x}, \mathbf{p}, t) + \sigma(\mathbf{x}, \mathbf{p}, t)^T \mathbf{u}(t)$$

Optimal control is

$$u_i(t) = \begin{cases} 1, & \text{if } \sigma_i > 0 \\ -1 & \text{if } \sigma_i < 0 \\ \text{unknown} & \text{if } \sigma_i = 0 \end{cases}$$

When $\sigma(\mathbf{x}, \mathbf{p}, t) = 0$ the control u has no effect on H

- the PMP fails: we have no information about the optimal control
- called singular, degenerate, irregular, or ambiguous control.

Singular control

If $\sigma(\mathbf{x}, \mathbf{p}, t) = 0$ only for isolated points there there is no problem. If $\sigma(\mathbf{x}, \mathbf{p}, t) = 0$ over an interval, then within the interval

$$\dot{\sigma}(\mathbf{x}, \mathbf{p}, t) = \ddot{\sigma}(\mathbf{x}, \mathbf{p}, t) = \ldots = 0$$

then singular control must be used.

- similar in nature to the CoV case where the functional is linear in y', and so we have a degenerate solution (see earlier lectures).

Singular control

Linear-autonomous time-minimization problem, where

$$H = \psi(\mathbf{x}, \mathbf{p}) + \sigma(\mathbf{x}, \mathbf{p}) u(t)$$

where $\sigma(\mathbf{x}, \mathbf{p}) = 0$ over some interval.

- autonomous problems implies $H = const$
- free-end time implies $H = 0$ for all $t \in [0, T]$
- So $\psi(\mathbf{x}, \mathbf{p}) = 0$ over the same interval as $\sigma(\mathbf{x}, \mathbf{p}) = 0$.
- Similarly for the kth order derivatives of ψ and σ
- Using the chain rule

$$\dot{\sigma}(\mathbf{x}, \mathbf{p}) = \frac{\partial \sigma}{\partial \mathbf{x}} \dot{\mathbf{x}} + \frac{\partial \sigma}{\partial \mathbf{p}} \dot{\mathbf{p}} = \frac{\partial \sigma}{\partial \mathbf{x}} f(\mathbf{x}, \mathbf{u}) + \frac{\partial \sigma}{\partial \mathbf{p}} \dot{\mathbf{p}} = 0$$

we may be able to solve for $\mathbf{u}$ (if not, increase k)

Singular control example

Minimize

$$F = \frac{1}{2}\int_0^T x_1^2\,dt$$

subject to

$$\begin{aligned}\dot{x}_1 &= x_2 + u\\ \dot{x}_2 &= -u\end{aligned}$$

where $|u| \le 1$ and T is unspecified.

The Hamiltonian is

$$H = -\frac{1}{2}x_1^2 + p_1(x_2+u) - p_2 u = -\frac{1}{2}x_1^2 + p_1 x_2 + (p_1 - p_2)u$$

which is linear in u, with switching function $\sigma = p_1 - p_2$.

Singular control example

Hamilton's equations

$$\frac{\partial H}{\partial p_i} = \frac{dx_i}{dt} \quad \text{and} \quad \frac{\partial H}{\partial x_i} = -\frac{dp_i}{dt}$$

Give the state equations and

$$\begin{aligned}\frac{\partial H}{\partial x_1} &= -x_1 = -\dot{p}_1\\ \frac{\partial H}{\partial x_2} &= p_1 = -\dot{p}_2\end{aligned}$$

The solution involves three cases

1. If the switching function $\sigma = p_1 - p_2 > 0$ then $u = 1$
2. If the switching function $\sigma = p_1 - p_2 < 0$ then $u = -1$
3. If the switching function $\sigma = p_1 - p_2 = 0$ then we have singular control

Singular control example

Case 1: $\sigma = p_1 - p_2 > 0$ and $u = 1$, so

$$\begin{aligned}\dot{x}_1 &= x_2 + 1\\ \dot{x}_2 &= -1\end{aligned}$$

which has solutions

$$\begin{aligned}x_1 &= -\frac{1}{2}t^2 + (c_1+1)t + c_2\\ x_2 &= -t + c_1\end{aligned}$$

so we can write

$$x_1 = -\frac{1}{2}x_2^2 - x_2 + c_4$$

where $c_4 = c_1(c_1+1) + c_2 - c_1^2/2$

Singular control example

Case 2: $\sigma = p_1 - p_2 < 0$ and $u = -1$, so

$$\begin{aligned}\dot{x}_1 &= x_2 - 1\\ \dot{x}_2 &= 1\end{aligned}$$

which has solutions

$$\begin{aligned}x_1 &= \frac{1}{2}t^2 + (c_1-1)t + c_2\\ x_2 &= t + c_1\end{aligned}$$

so we can write

$$x_1 = \frac{1}{2}x_2^2 - x_2 + c_3$$

where $c_3 = -c_1(c_1-1) + c_2 + c_1^2/2$

Singular control example

Case 3: singular as $\sigma = p_1 - p_2 = 0$

$$\begin{aligned}\sigma &= p_1 - p_2 \\ \dot{\sigma} &= \dot{p}_1 - \dot{p}_2 \\ &= x_1 + p_1 \\ &= 0\end{aligned}$$

Using this, and the fact that $p_1 - p_2 = 0$ in the Hamiltonian $H = -\frac{1}{2}x_1^2 + p_1 x_2 + (p_1 - p_2)u$, we get

$$H = -\frac{1}{2}x_1^2 + p_1 x_2 + (p_1 - p_2)u = -\frac{1}{2}x_1^2 - x_1 x_2$$

For autonomous problems, with free end time $H = 0$, so

$$x_1(x_2 + x_1/2) = 0$$

and hence, either $x_1 = 0$ or $x_1 + 2x_2 = 0$

Singular control example

The two solutions present two surfaces:

$$\begin{aligned}S_1 : &\quad x_1 = 0 \\ S_2 : &\quad x_1 + 2x_2 = 0\end{aligned}$$

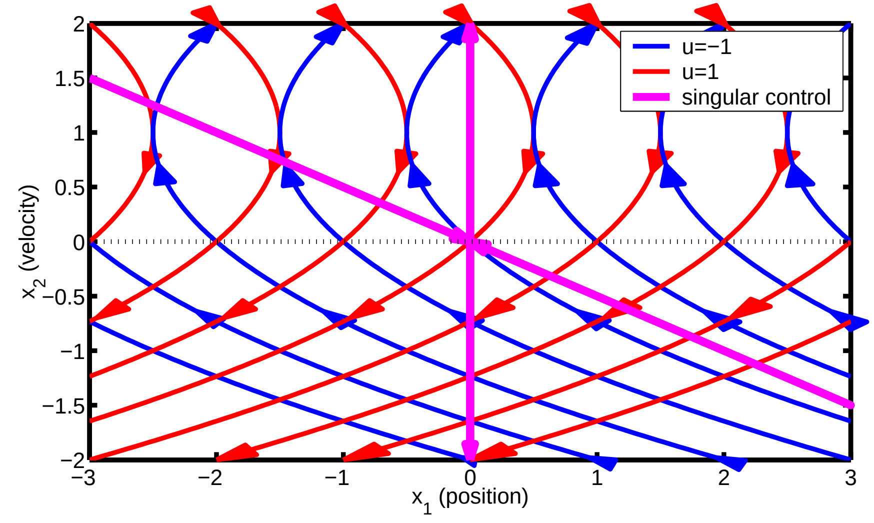
- on S_1 the derivative $\dot{x}_1 = 0$, and the state equation is $\dot{x}_1 = x_2 + u$, so $u = -x_2$.
- on S_2 the derivative $\dot{x}_2 = -\dot{x}_1/2$, and the state equations

$$\begin{aligned}\dot{x}_1 &= x_2 + u \\ \dot{x}_2 &= -u\end{aligned}$$

lead to $u = x_2$

Singular control example

Phase diagram



Singular control example

Phase diagram