Tutorial 2 Solutions

1. Find the extremals of the functionals below subject to the fixed end point conditions prescribed.

(a). $\displaystyle\int_0^{\pi/2} \left(y^2 + y'^2 - 2y\sin x\right)\, dx;\quad y(0) = 0,\, y(\pi/2) = 3/2.$

(b). $\displaystyle\int_1^2 \frac{y'^2\, dx}{x^3};\quad y(1) = 0,\, y(2) = 15.$

(c). $\displaystyle\int_0^2 \left(xy' + y'^2\right)\, dx;\quad y(0) = 1,\, y(2) = 0.$

Solutions:

(a) With

$$f(x, y, y') = y^2 + y'^2 - 2y\sin x\ ,$$

the Euler–Lagrange equation

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'}$$

becomes

$$2y - 2\sin x = \frac{d}{dx}\left(2y'\right)$$

or

$$y'' - y = -\sin x\ .$$

The complementary function (solution of the corresponding homogeneous equation) is

$$y = Ae^x + Be^{-x}.$$

For a particular integral, try $y = C\sin x$. Then

$$-C\sin x\ - C\sin x\ =\ -\sin x\ ,$$

so $C = 1/2$. Hence the G.S. to the E–L equation is

$$y = Ae^x + Be^{-x} + \frac{1}{2}\sin x\ .$$

The fixed end point $y(0) = 0$ provides $B = -A$, so

$$y = A\left(e^x - e^{-x}\right) + \frac{1}{2}\sin x\ .$$

The end point $y(\pi/2) = 3/2$ gives

$$y = A\left(e^{\pi/2} - e^{-\pi/2}\right) + \frac{1}{2} = \frac{3}{2}$$

and so

$$A = \frac{1}{e^{\pi/2} - e^{-\pi/2}}\ .$$

Therefore

$$y = \frac{e^x - e^{-x}}{e^{\pi/2} - e^{-\pi/2}} + \frac{1}{2}\sin x = \frac{\sinh(x)}{\sinh(\pi/2)} + \frac{1}{2}\sin x.$$

(b) Since $f(x, y, y') = y'^2/x^3$ doesn't involve y explicitly, a first integral to E–L is $\partial f/\partial y' = \text{const.}$ or

$$\frac{y'}{x^3} = C.$$

Integration yields

$$y = \frac{Cx^4}{4} + D.$$

Since $y(1) = 0$, we have $\frac{C}{4} + D = 0$, so

$$y = \frac{C}{4}\left(x^4 - 1\right)$$

and $y(2) = 15$ gives

$$15 = \frac{C}{4}\left(16 - 1\right).$$

Hence $C/4 = 1$ and

$$y = x^4 - 1.$$

(c) Since $f(x, y, y') = xy' + y'^2$ doesn't involve y explicitly, a first integral to E–L is $\partial f/\partial y' = \text{const.}$ or

$$x + 2y' = C.$$

Integration provides

$$y = \frac{C}{2}x - \frac{x^2}{4} + D.$$

As $y(0) = 1$, we have $D = 1$ and

$$y = \frac{C}{2}x - \frac{x^2}{4} + 1,$$

and $y(2) = 0$ now gives

$$0 = C - 1 + 1$$

or $C = 0$. Hence

$$y = 1 - \frac{x^2}{4}\ .$$

2. Can light bend along a circular arc, purely through refraction? Explain your answer.

Solutions:

Fermat's principle of least time (1661) states that a beam of light propagated in a medium having a velocity of light gradient, i.e. a refractive index gradient, travels along a path between two points that takes a minimum time.

Choose $c(y)$, the speed of light at point y, such that time elapsed by passage of light between two fixed points is a minimum on these arcs, with respect to all possible paths connecting the two points.

The total time elapsed along a path is

$$T\{y(x)\} = \int_A^B dt = \int_A^B \frac{1}{c(y)}\, ds = \int_{x_0}^{x_1} \frac{\sqrt{1+y'^2}}{c(y)}\, ds$$

Here f is not explicitly dependent of x, so we can form

$$H(y, y') = y' \frac{\partial f}{\partial y'} - f = \frac{y'^2}{c(y)\sqrt{1+y'^2}} - \frac{\sqrt{1+y'^2}}{c(y)} = k_1 = const$$

Multiplying by $\sqrt{1+y'^2}$ we get

$$\frac{y'^2}{c(y)} - \frac{(1+y'^2)}{c(y)} = k_1\sqrt{1+y'^2}$$

and so

$$c(y) = -\frac{1}{k_1\sqrt{1+y'^2}}$$

Now if we require the light rays to be on circular arcs, then we need to define a circle that lies on the start and end point. To make life easy, we shall choose end points that lie on the circle, with radius R, and center (a, b), e.g., $(x_0, y_0) = (a - R, b)$ and $(x_1, y_1) = (a + R, b)$, then we get

$$(x-a)^2 + (y-b)^2 = R^2$$

Differentiating with respect to x gives

$$2(x-a) + 2(y-b)y' = 0$$

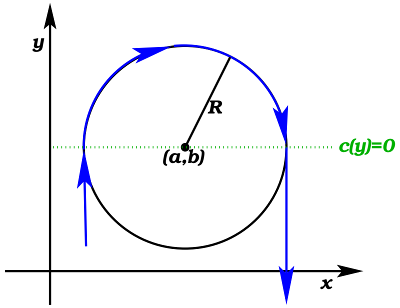
from which we derive

$$\begin{aligned}
y' &= -\frac{x-a}{y-b} \\
1 + y'^2 &= 1 + \frac{(x-a)^2}{(y-b)^2} \\
&= \frac{(x-a)^2 + (y-b)^2}{(y-b)^2} \\
&= \frac{R^2}{(y-b)^2} \\
\sqrt{1+y'^2} &= \pm\frac{R}{(y-b)}
\end{aligned}$$

So we can choose

$$c(y) = \frac{y-b}{k_1 R}$$

with the result that light traveling from point $(a - R, b)$ to $(a + R, b)$ will traverse a circular arc.



It may seem unreasonable to suggest that the speed of light can vary continuously, but it does exactly that. For instance, the speed of light in air, as a function of temperatire, pressure, and wavelength is given (D. R. Linde. CRC Handbook of Chemistry and Physics. CRC Press, 1995).

$$(n-1) \cdot 10^8 = 8342.13 + 2406030(130 - \sigma^2)^{-1} + 15997(38.9 - \sigma^2)^{-1}$$

where

$$\begin{aligned}
\sigma &= 1/\lambda, \text{ where } \lambda = \text{ wavelength in } \mu m, \\
T &= \text{Temperature in degrees C}, \\
p &= \text{pressure in } N/m^2.
\end{aligned}$$

and if $T \neq 15$ degrees, or $p \neq 101.325\, kP$ then $(n-1)$ above is multiplied by

$$\frac{p(1+p(61.3-T)\cdot 10^{-10})}{96095.4(1+0.003661T)}.$$

We see this type of affect at work above a road heated by the sun – the air near the roadway is hotter, and hence light is bent. The results are the mirages of "water" common on Australian road. It isn't really water you can see, but rather the refracted light from the blue sky.

We can derive other formulae for other types of EM radiation, and this likewise causes light to bend, e.g., the Ionosphere's "reflection" of radio-waves is actually a similar type of refraction.

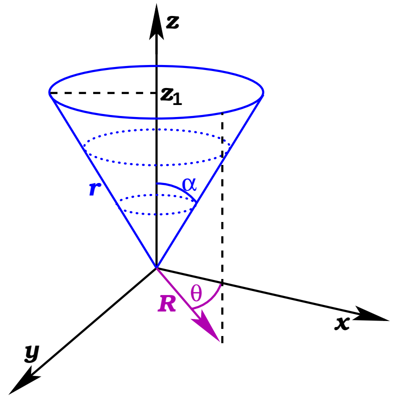
The circular solution is slightly strange. After all, there is no asymmetry in the problem, so why would light, travelling directly upwards at $(x_0, y_0) = (a-R, b)$ go to the right, instead of the left? The answer lies in two details we haven't considered carefully. First, the question implicitly asked using the CoV formalism is not "What is the arc light takes from the start point?" It is actually, "What would the path be between A and B?"

Secondly, note that in the formulation above, the speed of light at $y = b$ is zero, so we can't actually start at the point where we are travelling vertically upwards (because light would not be moving at this point).

Thirdly, in support of the last point, in formulating the problem we assumed we could write y and y' as functions of x, which is not the case if $y' = \infty$.

Finally, what is k_1 here?

3. Find the geodesics of a right circular cone? Also find shortest-path transversals from the top of the cone (a circle at height z_1) to any point on the cone.



Solutions:

Methods for solution: consider the general Euler-Lagrange equations of geodesics from Lecture 6, or adding an extra constraint describing the surface as in Lecture 13, but we shall use a direct approach.
Define the right-circular cone S to have its axis coinciding with the z-axis, and let $\alpha = const$ be half the angle at the vertex.
First change to alternative spherical polar coordinates (r, θ, ϕ), (Physicists use this form).

$$\begin{aligned} x &= r\cos(\theta)\sin(\phi) \\ y &= r\sin(\theta)\sin(\phi) \\ z &= r\cos(\phi) \end{aligned}$$

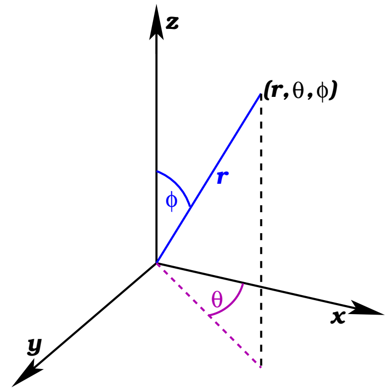
In the coordinates, the cone is represented by the constraint

$$\phi = \alpha = const.$$

And a curve γ is given by

$$\begin{aligned} \phi &= \alpha \\ \theta &= \theta_\gamma(r) \end{aligned}$$

Spherical Polar Coordinates



The length of a curve between point A and B is

$$L\{\theta(r)\} = \int_A^B ds = \int_A^B \frac{ds}{dr}\,dr = \int_{r_0}^{r_1} \sqrt{1 + r^2\sin^2(\alpha)\theta'^2}\,dr$$

because (from the Chain Rule)

$$\begin{aligned}
dx &= \frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial\phi}d\phi + \frac{\partial x}{\partial r}dr &&= -r\sin(\theta)\sin(\phi)d\theta + r\cos(\theta)\cos(\phi)d\phi + \cos(\theta)\sin(\phi)dr \\
dy &= \frac{\partial y}{\partial\theta}d\theta + \frac{\partial y}{\partial\phi}d\phi + \frac{\partial y}{\partial r}dr &&= r\cos(\theta)\sin(\phi)d\theta + r\sin(\theta)\cos(\phi)d\phi + \sin(\theta)\sin(\phi)dr \\
dz &= \frac{\partial z}{\partial\theta}d\theta + \frac{\partial z}{\partial\phi}d\phi + \frac{\partial z}{\partial r}dr &&= -r\sin(\phi)d\phi + \cos(\phi)dr
\end{aligned}$$

but note that for the cone, $\phi = const$, so $d\phi = 0$ and so

$$\begin{aligned}
ds^2 &= dx^2 + dy^2 + dz^2 \\
&= r^2\sin^2(\theta)\sin^2(\phi)d\theta^2 + \cos^2(\theta)\sin^2(\phi)dr^2 - 2r\sin(\theta)\sin(\phi)\cos(\theta)\sin(\phi)d\theta\,dr \\
&\quad + r^2\cos^2(\theta)\sin^2(\phi)d\theta^2 + \sin^2(\theta)\sin^2(\phi)dr^2 + 2r\cos(\theta)\sin(\phi)\sin(\theta)\sin(\phi)d\theta\,dr \\
&\quad + \cos^2(\phi)dr^2 \\
&= \left(\cos^2(\theta)\sin^2(\phi) + \sin^2(\theta)\sin^2(\phi) + \cos^2(\phi)\right)dr^2 + r^2\left(\sin^2(\theta)\sin^2(\phi) + \cos^2(\theta)\sin^2(\phi)\right) \\
&= dr^2 + r^2\sin^2(\phi)d\theta^2 \\
\frac{ds}{dr} &= \sqrt{1 + r^2\sin^2(\phi)\left(\frac{d\theta}{dr}\right)^2}
\end{aligned}$$

where $\phi = \alpha$. We wish to find the curve γ which minimizes $L\{\theta\}$, so we use the Euler-Lagrange equation (where here $f(r,\theta')$ is independent of θ, so the Euler-Lagrange equation

$$f_\theta - \frac{d}{dr}f_{\theta'} = 0 \quad\Rightarrow\quad \frac{d}{dr}f_{\theta'} = 0$$

and therefore

$$\begin{aligned}
f_{\theta'} \;=\; && \frac{r^2\sin^2(\alpha)\theta'}{\sqrt{1+r^2\sin^2(\alpha)\theta'^2}} &= k_1 = const \\
\Rightarrow && r^2\sin^2(\alpha)\theta' &= k_1\sqrt{1+r^2\sin^2(\alpha)\theta'^2} \\
\Rightarrow && r^4\sin^4(\alpha)\theta'^2 &= k_1^2\left(1+r^2\sin^2(\alpha)\theta'^2\right)^2 \\
\Rightarrow && \left(r^4\sin^4(\alpha) - k_1^2r^2\sin^2(\alpha)\right)\theta'^2 &= k_1^2 \\
\Rightarrow && r^2\sin^2(\alpha)\left(r^2\sin^2(\alpha) - k_1^2\right)\theta'^2 &= k_1^2 \\
\Rightarrow && \theta'^2 &= \frac{k_1^2}{r^2\sin^2(\alpha)\left(r^2\sin^2(\alpha)-k_1^2\right)} \\
\Rightarrow && \frac{d\theta}{dr} &= \frac{k_1}{r\sin(\alpha)\sqrt{r^2\sin^2(\alpha)-k_1^2}} \\
\Rightarrow && \theta + k_2 &= \frac{k_1}{\sin^2(\alpha)}\int\frac{dr}{r\sqrt{r^2 - k_1^2/\sin^2(\alpha)}} \\
\Rightarrow && \theta + k_2 &= \frac{1}{\sin(\alpha)}\sec^{-1}\left(\frac{r\sin(\alpha)}{k_1}\right)
\end{aligned}$$

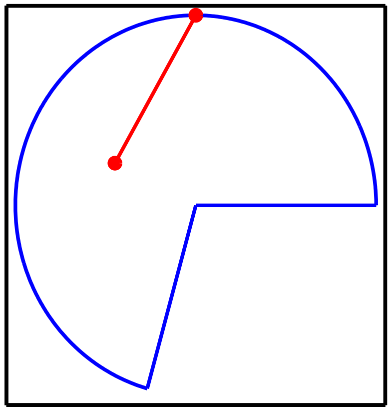
Changing the constants $A = k_1/\sin(\alpha)$ and $B = k_2\sin(\alpha)$ we get

$$r = A\sec\left[\theta\sin(\alpha) + B\right]$$

Note that, the above solution may admit multiple curves, each of which may be a local minimum, but not a global minimum!

Another approach to the solution is transform the surface to a more familiar surface (see Lecture 7), and exploit invariance of the E-L equations under the transformation, e.g., unwrap the cone, to get a segment of a flat circle. Geodesics on the circle are straight lines, and so geodesics on the cone may be obtained by transforming the straight lines on the circle segment onto the surface of a cone. This approach is very similar to that above, except we rotate the coordinates so that we start with geodesics of the form

$$X = c_1$$



where (X,Y) are coodinates on flattened cone, and note that to get to coordinates on the surface of the rolled up cone we need take polar coordinates in the plane, but we can rotate these coordinates so that the line is parallel to the y-axis, e.g. $X = c_1 = const$.

$$\begin{aligned}
R &= \sqrt{X^2+Y^2} \\
&= \sqrt{c_1^2+Y^2} \\
\lambda &= \tan^{-1}(Y/X) = \tan^{-1}(Y/c_1) \\
&= \tan^{-1}(\sqrt{R^2-c_1^2}/c_1)
\end{aligned}$$

and convert these to spherical-polar coordinates on the cone. The transform is

$$\begin{aligned}
r &= R \\
\phi &= \alpha \\
\theta &= \lambda/\sin(\alpha) + \beta
\end{aligned}$$

where β is the amount by which we rotated the original (X,Y) axis.

The equation for θ comes from the fact that as we roll up the cone, the outer circumference of the circular part must map to the top circle of the cone, and so an angle λ in polar coordiates in the plane, will be reduced by a factor of the ratio of the circumferences of the circles, or $1/\sin(\alpha)$.

Now note that

$$\tan^{-1}(x) = \begin{cases} \sec^{-1}(\sqrt{x^2+1}), & \text{if } x > 0 \\ -\sec^{-1}(\sqrt{x^2+1}), & \text{if } x < 0 \end{cases}$$

so we can write (for positive arguments)

$$\lambda = \tan^{-1}\left(\frac{\sqrt{R^2 - c_1^2}}{c_1}\right) = \tan^{-1}\left(\sqrt{R^2/c_1^2 - 1}\right) = \sec^{-1}(R/c_1)$$

which results once again in

$$\theta - \beta = \frac{1}{\sin(\alpha)}\sec^{-1}(R/c_1)$$

where in this case, the constants are immediately defined by the coordinates of the start and end-points in the (X, Y) plane.

4. The Beltrami identity states that the extremal function of the integral

$$I\{u\} = \int_a^b L(x, u, u')\,dx$$

satisfy the differential equation

$$\frac{d}{dx}\left(L - u'\frac{\partial L}{\partial u'}\right) - \frac{\partial L}{\partial x} = 0.$$

Please prove the identity using the Euler-Lagrange equations and the chain rule. Note that as a special case, when L does not depend on x, we get the equation for the autonomous case, i.e., $H = const$.

Solutions: Take the derivative of L with respect to x and apply the chain rule, i.e.,

$$\begin{aligned}
\frac{dL}{dx} &= \frac{\partial L}{\partial x}\frac{dx}{dx} + \frac{\partial L}{\partial u}\frac{du}{dx} + \frac{\partial L}{\partial u'}\frac{du'}{dx} \\
&= \frac{\partial L}{\partial x} + \frac{\partial L}{\partial u}u' + \frac{\partial L}{\partial u'}u'' \\
\frac{\partial L}{\partial u}u' &= \frac{dL}{dx} - \frac{\partial L}{\partial x} - \frac{\partial L}{\partial u'}u'' \qquad (1)
\end{aligned}$$

Multiply the Euler-Lagrange equations by u' and we get

$$u'\frac{d}{dx}\frac{\partial L}{\partial u'} - u'\frac{\partial L}{\partial u} = 0$$

Substitute (1) into the Euler-Lagrange equation

$$\begin{aligned}
u'\frac{d}{dx}\frac{\partial L}{\partial u'} - \frac{dL}{dx} + \frac{\partial L}{\partial x} + \frac{\partial L}{\partial u'}u'' &= 0 \\
\frac{d}{dx}\left[u'\frac{\partial L}{\partial u'} - L\right] + \frac{\partial L}{\partial x} &= 0
\end{aligned}$$

which simply rearranges to give the Beltrami identity.

Notice that when L does not depend on x, then

$$\frac{\partial L}{\partial x} = 0,$$

and so the identity reduces to

$$H = u'\frac{\partial L}{\partial u'} - L = const.$$

5. Newton's aerodynamic problem (the problem of finding the surface of revolution that minimizes drag) is often approximated by assuming the shape is long and thin, so that y' is large (and negative). In this case we can approximate

$$\frac{1}{1+y'^2} \simeq \frac{1}{y'^2}$$

and the functional of interest by

$$F\{y\} \simeq \int_0^R \frac{x}{y'^2}\,dx,$$

Derive the shape that arise from minimizing this functional.

Solution: The Euler-Lagrange equations are

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = -2\frac{d}{dx}\frac{x}{y'^3} = 0.$$

So we get

$$y' = c_1 x^{1/3}$$

Integrating and calculating the constants we get

$$y = -L(x/R)^{4/3} + L.$$

We can calculate the drag for the approximate functional as

$$F\{y\} = \int_0^R \frac{x}{y'^2}\,dx = \frac{3^2 R^{8/3}}{4^2 L^2}\int_0^R x^{1/3} = \frac{3^3 R^{8/3}}{4^3 L^2}\left[x^{4/3}\right]_0^R = \frac{3^3 R^4}{4^3 L^2}.$$

However, we must remember that the actual functional is $F\{y\} = \int_0^R \frac{x}{1+y'^2}\,dx$, and the above is an approximation, which is not valid when the nose cone becomes blunt.

The function describing the nose-cone shape is usually plotted with the x-axis as the centerline of the nose-cone, so

$$g(x) = R(x/L)^{3/4}.$$

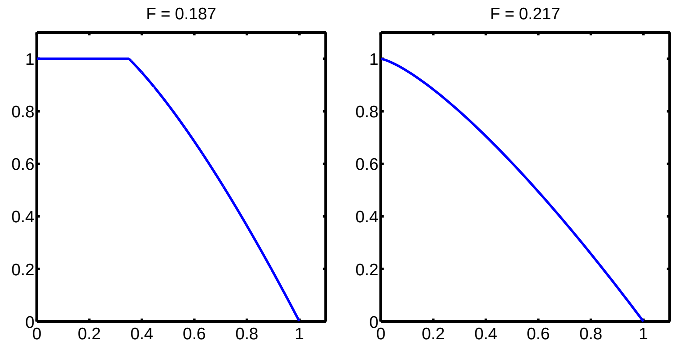
This solution is one of the standard approximations to the aerodynamic nose-cone problem, oddly enough called a "parabolic" nose cone, from the family of

$$g(x) = R(x/L)^{\alpha}, \quad \alpha \in [0, 1.0].$$

The resulting power-law shape is appealing because its easy to draw, and easy to calculate the functional. Note that the family includes the blunt faced cylinder and the cone as special cases as well as the solution above.

The following figure compares its optimality when used in the unapproximated functional. The optimal curve is on the left, and we can see it has a much lower

resistance. They would be much close if L/R were bigger (and hence slopes were larger), but even so we can see the value of our exact solution.



The shape doesn't satisfy a common requirement for such nose cones that they be tangent to the base cylinder where they touch, but we haven't really considered that constraint yet (and Newton's solution doesn't meet this criteria either).