Tutorial 3 Solutions

1. **Higher-order derivatives:** Go through the steps of deriving the Euler-Poisson equation for a functional containing derivatives or order three, i.e.,

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y', y^{(2)}, y^{(3)}), dx.$$

Solutions: Let $F : C^2[x_0, x_1] \to I\!R$ be a functional of the form

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y', y^{(2)}, y^{(3)})\, dx,$$

where f has continuous partial derivatives of second order with respect to x, y, y', $y^{(2)}$, and $y^{(3)}$, and $x_0 < x_1$. As before, the necessary condition for the extremum is that the first variation be zero, e.g.

$$\delta F(\eta, y) = 0.$$

As in lectures we perturb y to get $\hat{y} = y + \varepsilon\eta$ and apply Taylor's theorem to derive

$$\begin{aligned} &f(x, y+\varepsilon\eta, y'+\varepsilon\eta', y^{(2)}+\varepsilon\eta^{(2)}, y^{(3)}+\varepsilon\eta^{(3)}) = \\ &\qquad f(x, y, y', y^{(2)}, y^{(3)}) + \varepsilon\left[\eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'} + \eta^{(2)}\frac{\partial f}{\partial y^{(2)}} + \eta^{(3)}\frac{\partial f}{\partial y^{(3)}}\right] + O(\varepsilon^2) \end{aligned}$$

and hence

$$F\{y+\varepsilon\eta\} = \int_{x_0}^{x_1} f(x, y, y', y^{(2)}, y^{(3)}) + \varepsilon\left[\eta\frac{\partial f}{\partial y} + \eta^{(2)}\frac{\partial f}{\partial y^{(2)}} + \eta^{(3)}\frac{\partial f}{\partial y^{(3)}}\right] dx + O(\varepsilon^2)$$

So, now the first variation will be given by

$$\begin{aligned}
\delta F(\eta, y) &= \lim_{\varepsilon\to 0}\frac{F\{y+\varepsilon\eta\} - F\{y\}}{\varepsilon} \\
&= \int_{x_0}^{x_1}\left[\eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'} + \eta^{(2)}\frac{\partial f}{\partial y^{(2)}} + \eta^{(3)}\frac{\partial f}{\partial y^{(3)}}\right] dx \\
&= \left[\eta\frac{\partial f}{\partial y'}\right]_{x_0}^{x_1} + \left[\eta'\frac{\partial f}{\partial y^{(2)}}\right]_{x_0}^{x_1} + \left[\eta^{(2)}\frac{\partial f}{\partial y^{(3)}}\right]_{x_0}^{x_1} + \int_{x_0}^{x_1}\left[\eta\frac{\partial f}{\partial y} - \eta\frac{d}{dx}\frac{\partial f}{\partial y'} - \eta'\frac{d}{dx}\frac{\partial f}{\partial y^{(2)}} - \eta^{(\ldots}\right. \\
&= \left[\eta\frac{\partial f}{\partial y'}\right]_{x_0}^{x_1} + \left[\eta'\frac{\partial f}{\partial y^{(2)}}\right]_{x_0}^{x_1} + \left[\eta^{(2)}\frac{\partial f}{\partial y^{(3)}}\right]_{x_0}^{x_1} - \left[\eta\frac{d}{dx}\frac{\partial f}{\partial y^{(2)}}\right]_{x_0}^{x_1} - \left[\eta'\frac{d}{dx}\frac{\partial f}{\partial y^{(3)}}\right]_{x_0}^{x_1} \\
&\quad + \int_{x_0}^{x_1}\left[\eta\frac{\partial f}{\partial y} - \eta\frac{d}{dx}\frac{\partial f}{\partial y'} + \eta\frac{d^2}{dx^2}\frac{\partial f}{\partial y^{(2)}} + \eta'\frac{d^2}{dx^2}\frac{\partial f}{\partial y^{(3)}}\right] dx \\
&= \left[\eta\frac{\partial f}{\partial y'}\right]_{x_0}^{x_1} + \left[\eta'\frac{\partial f}{\partial y^{(2)}}\right]_{x_0}^{x_1} + \left[\eta^{(2)}\frac{\partial f}{\partial y^{(3)}}\right]_{x_0}^{x_1} - \left[\eta\frac{d}{dx}\frac{\partial f}{\partial y^{(2)}}\right]_{x_0}^{x_1} - \left[\eta'\frac{d}{dx}\frac{\partial f}{\partial y^{(3)}}\right]_{x_0}^{x_1} + \left[\eta\frac{d}{d\ldots}\right. \\
&\quad + \int_{x_0}^{x_1}\eta\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y^{(2)}} - \frac{d^3}{dx^3}\frac{\partial f}{\partial y^{(3)}}\right] dx
\end{aligned}$$

Given fixed-end point conditions

$$\begin{aligned} y(x_0) &= y_0 & y(x_1) &= y_1 \\ y'(x_0) &= y_0' & y'(x_1) &= y_1' \\ y^{(2)}(x_0) &= y_0^{(2)} & y^{(2)}(x_1) &= y_1^{(2)} \end{aligned}$$

we have

$$\begin{aligned} \eta(x_0) &= 0 & \eta(x_1) &= 0 \\ \eta'(x_0) &= 0 & \eta'(x_1) &= 0 \\ \eta^{(2)}(x_0) &= 0 & \eta^{(2)}(x_1) &= 0 \end{aligned}$$

Which gives

$$\delta F(\eta, y) = \int_{x_0}^{x_1}\eta\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y^{(2)}} - \frac{d^3}{dx^3}\frac{\partial f}{\partial y^{(3)}}\right] dx$$

$\delta F(\eta, y) = 0$ for arbitrary η satisfying the boundary conditions, so the result is the 6th order Euler-Poisson equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y^{(2)}} - \frac{d^3}{dx^3}\frac{\partial f}{\partial y^{(3)}} = 0.$$

2. **Multiple dependent variables:** calculate the form of geodesics in N-dimensional Euclidean space.

Solution: We can parameterize a curve in N dimensions by $(q_1(t), q_2(t), \ldots, q_n(t))$, where the q_i are the location co-ordinates.

The objective is to minimize the distance along a geodesic, and so the functional of interest is

$$F\{\mathbf{q}\} = \int_{t_0}^{t_1} ds,$$

but this is not in a suitable form for our derivation. However, we can rewrite using the fact that the length of the line segment from t to $t + \delta s$ is the length of a line segment from $\mathbf{q}$ to $\mathbf{q} + \delta\mathbf{q}$ which is approximately

$$\begin{aligned} \delta s &= \sqrt{\sum_i \delta q_i^2} \\ &= \delta t\sqrt{\sum_i \frac{\delta q_i^2}{\delta t^2}}. \end{aligned}$$

Taking small δt and integrating we can write the path length as

$$F\{y\} = \int_{t_0}^{t_1} ds = \int_{t_0}^{t_1}\sqrt{\sum_i \dot{q}_i^2}\, dt.$$

We get one Euler-Lagrange equation for eqach co-ordinate, and the objective has no t or q_i terms, and so

$$\frac{d}{dx}\frac{\partial f}{\partial \dot{q}_i} = 0$$

for all i. Simplifying this we get

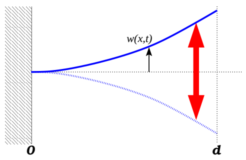
$$\ddot{q}_i = 0,$$

and so

$$q_i = c_i t + d_i,$$

for some set of constants c_i and d_i. This is the parametric form of a straight line, so the geodesics in N-dimensions are straight lines.

3. Multiple independent variables: take a beam length d with flexural rigidity κ and density per unit length ρ, fixed and clamped at one end, and derive the motion of this beam when one end is held (displaced from its equilibrium position) and then released suddenly (see figure). NB: $\kappa = EI$ where E is the Young's modulus, and I is the area of moment of inertia of the beam.



Hints:

- Assume the beam is thin, and it is not bent too far.
- Ignore gravitational potential for the purpose of solving this problem, and assume deflections are small enough that the beam can be modelled by considering only vertical deflections, so that we can see the notation that the displacement of the beam at distance x from the clamp and time t is $w(x,t)$. We will use w_x and w_t as shorthand for the relevant partial derivatives.
- The boundary conditions for $w(x,t)$ will be

$$\begin{aligned} w(0,t) &= 0, && \text{because the left end point is fixed} \\ w_x(0,t) &= 0, && \text{because the left end point is clamped} \\ w_t(x,0) &= 0, && \text{because at the start the beam is stationary} \\ w_{xx}(d,t) &= 0, && \text{because the free end point has zero bending moment} \\ w_{xxx}(d,t) &= 0, && \text{because the free end point has zero shearing force} \end{aligned}$$

where the shape $y(x)$ is determined by the force being applied to the beam before it is released.

- You may assume the solution is separable, i.e., that $w(x,t) = h(x)g(t)$, i.e., that we are looking for a "normal mode" of vibration in which all the components of the beam move with the same frequency and in phase.

Solutions: Ignoring gravity, the two components we need to calculate are the elastic potential, and the kinetic energy at each point in time. These are given by

$$\begin{aligned} T &= \int_0^d \frac{\rho}{2}\left(\frac{\partial w}{\partial t}\right)^2 dx \\ V &= \int_0^d \frac{\kappa}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx \end{aligned}$$

Hamilton's principle leads us to look for extremal curves of

$$F\{w\} = \int_{t_0}^{t_1} T - V\, dt = \int_{t_0}^{t_1}\int_0^d \frac{\rho}{2}w_t^2 - \frac{\kappa}{2}w_{xx}^2\, dx\, dt.$$

Note that this involves both multiple indendent variables, and higher order derivatives, but the extension of the Euler-Lagrange equations should be natural. Ignoring zero terms, the E-L equations take the form

$$-\frac{\partial}{\partial t}\frac{\partial f}{\partial w_t} + \frac{\partial^2}{\partial x^2}\frac{\partial f}{\partial w_{xx}} = 0.$$

Taking $f = \frac{\rho}{2}w_t^2 - \frac{\kappa}{2}w_{xx}^2$ as in the integral we get the Euler-Lagrange-Poisson equation to be

$$\kappa w_{xxxx} + \rho w_{tt} = 0$$

Now take the separable solution $w(x,t) = h(x)g(t)$ and we get

$$\kappa h^{(4)}(x)g(t) + \rho h(x)g''(t) = 0,$$

Now, we can rearrange this to get

$$a^2\frac{h^{(4)}(x)}{h(x)} = -\frac{g''(t)}{g(t)},$$

where we take $a^2 = \kappa/\rho$. As the LHS is constant WRT to t, and the RHS is constant WRT to x, they must both equal a constant.

We start with the RHS, and set the constant to be ω^2 so the resulting DE is

$$g'' + \omega^2 g = 0.$$

The solution to this equation is

$$g(t) = A\cos(\omega t) + B\sin(\omega t),$$

but at $t = 0$, the beam is held at rest, so

$$g(t) = A\cos(\omega t).$$

Clearly ω is the frequency of vibration.

Now take the LHS of the equation and rearrange and we get

$$h^{(4)}(x) - \frac{\omega^2}{a^2} h(x) = 0.$$

The solutions of this DE clearly depend on the forth roots or $k^4 = \omega^2/a^2$, and hence can be written as a linear combination of $\sin$, $\cos$, $\sinh$ and $\cosh$, but it is convenient to write them in the following form:

$$\begin{aligned} h(x) &= c_1\left[\sin(kx)+\sinh(kx)\right] + c_2\left[\sin(kx)-\sinh(kx)\right] \\ &\quad + c_3\left[\cos(kx)+\cosh(kx)\right] + c_4\left[\cos(kx)-\cosh(kx)\right]. \end{aligned}$$

The derivatives of $h(x)$ are

$$\begin{aligned} \frac{h'(x)}{k} &= c_1\left[\cos(kx)+\cosh(kx)\right] + c_2\left[\cos(kx)-\cosh(kx)\right] \\ &\quad + c_3\left[-\sin(kx)+\sinh(kx)\right] + c_4\left[-\sin(kx)-\sinh(kx)\right] \\ \frac{h''(x)}{k^2} &= c_1\left[-\sin(kx)+\sinh(kx)\right] + c_2\left[-\sin(kx)-\sinh(kx)\right] \\ &\quad + c_3\left[-\cos(kx)+\cosh(kx)\right] + c_4\left[-\cos(kx)-\cosh(kx)\right] \\ \frac{h'''(x)}{k^3} &= c_1\left[-\cos(kx)+\cosh(kx)\right] + c_2\left[-\cos(kx)-\cosh(kx)\right] \\ &\quad + c_3\left[\sin(kx)+\sinh(kx)\right] + c_4\left[\sin(kx)-\sinh(kx)\right] \end{aligned}$$

Now, from the end-point equations and $\cos(0) = \cosh(0) = 1$ and $\sin(0) = \sinh(0) = 0$ we get

$$\begin{aligned} w(0,t) &= 0, \;\Rightarrow\; h(0) = 0 \;\Rightarrow\; c_3 = 0 \\ w_x(0,t) &= 0, \;\Rightarrow\; h'(0) = 0 \;\Rightarrow\; c_1 = 0 \\ w_{xx}(d,t) &= 0, \;\Rightarrow\; h''(d) = 0 \\ w_{xxx}(d,t) &= 0, \;\Rightarrow\; h'''(d) = 0 \end{aligned}$$

where we must do a little furher work to refine the latter two. They give

$$\begin{aligned} c_2\left[-\sin(kd)-\sinh(kd)\right] + c_4\left[-\cos(kd)-\cosh(kd)\right] &= 0 \\ c_2\left[-\cos(kd)-\cosh(kd)\right] + c_4\left[\sin(kd)-\sinh(kd)\right] &= 0 \end{aligned}$$

We solve by multiplying the top equation by $[-\cos(kd) - \cosh(kd)]$, and the bottom by $[-\sin(kd) - \sinh(kd)]$ and subtracting (assuming $c_4\ neq 0$) to get

$$\begin{aligned} \left[-\cos(kd)-\cosh(kd)\right]\left[-\cos(kd)-\cosh(kd)\right] & \\ -\left[\sin(kd)-\sinh(kd)\right]\left[-\sin(kd)-\sinh(kd)\right] &= 0 \\ \cos^2+\cosh^2 - 2\cos\cosh + \sin^2 - \sinh^2 &= 0 \\ \cos^2+\sin^2+\cosh^2-\sinh^2 - 2\cos\cosh + \sin^2 &= 0 \\ \cos(kd)\cosh(kd) &= -1 \end{aligned}$$

When we solve this we find multiple values of k_n that satisfy the equation, each of which corresponds to a different mode of vibration. Note also that the second equations implies that

$$c_4 = c_2 \frac{\cos(kd)+\cosh(kd)}{\sin(kd)-\sinh(kd)},$$

so the solution for a particular mode of vibration is

$$h_n(x) = c_n\left[\left(\sin(k_n x) - \sinh(k_n x)\right) + \frac{\cos(k_n d)+\cosh(k_n d)}{\sin(k_n d)-\sinh(k_n d)}\left(\cos(k_n x)-\cosh(k_n x)\right).\right].$$

Once we know k_n, we can use a linear combination of vibration modes to match the initial state of the bent beam, and thence calculate its behaviour.

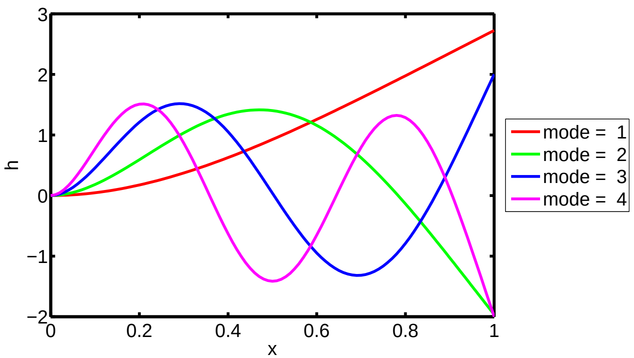
The following shows a numerically calculated table of the values of $k_n d$ (obviously to get k divide by d). We can also use the fact that $a^2 = \kappa/\rho$ and $k^4 = \omega^2/a^2$ to note that

$$\omega = \frac{k_n^2}{2\pi}\sqrt{\frac{\kappa}{\rho}},$$

measured in Hz (the division by 2π converts from radians per second to Hz).

mode	kd
1	1.87510407
2	4.69409113
3	7.85475744
4	10.99554073
5	14.13716839
6	17.27875953
7	20.42035225

The figure below shows the shape of the first few vibration modes. These modes don't have the conventional harmonic structure of a musical instrument, and so the sound of such a beam vibrating (e.g., a ruler vibrated on a bench) sounds dull and unmusical.



4. Ritz's Method and Higher Order Derivatives: Use Ritz's method to find an approximate solution to minimize the

$$J\{y\} = \int_0^{2\pi} y'^2 + \lambda^2 y^2 \, dx,$$

where $y(0) = 1$ and $y(2\pi) = 1$ and λ is a positive integer. Use the trial functions

$$\phi_n(x) = \cos(nx).$$

Compare your solution to one found directly from the Euler-Lagrange equations.

Solutions: The test functions $\phi_n(x) = \cos(nx)$ satisfy the boundary conditions. We take the approximation

$$y_N = \sum_{i=1}^N \alpha_i \phi_i(x) = \sum_{i=1}^N \alpha_i \cos(ix).$$

The derivative is

$$y_N' = -\sum_{i=1}^N \alpha_i i \sin(ix).$$

Substituting into the above we get

$$\begin{aligned} J\{y\} &= \int_0^{2\pi} y'^2 + \lambda^2 y^2 \, dx \\ &= \int_0^{2\pi} \sum_{i,j=1}^N \alpha_i i \alpha_j j \sin(ix)\sin(jx) - \lambda^2 \sum_{i,j=1}^N \alpha_i \alpha_j \cos(ix)\cos(jx)\, dx \\ &= \sum_{i,j=1}^N \alpha_i i \alpha_j j \int_0^{2\pi} \sin(ix)\sin(jx)\, dx - \lambda^2 \sum_{i,j=1}^N \alpha_i \alpha_j \int_0^{2\pi} \cos(ix)\cos(jx)\, dx \end{aligned}$$

Now

$$\begin{aligned} \int_0^{2\pi} \cos(nx)\, \cos(mx)\, dx &= \pi \delta_{mn} \\ \int_0^{2\pi} \sin(nx)\, \sin(mx)\, dx &= \pi \delta_{mn} \\ \int_0^{2\pi} \sin(nx)\, \cos(mx)\, dx &= 0 \end{aligned}$$

So

$$J\{y\} = \pi \sum_{i=1}^N \alpha_i^2 (i^2 - \lambda^2)$$

Taking the derivative we get

$$\begin{aligned} \frac{\partial J}{\partial \alpha_i} &= 2\pi \alpha_i (i^2 - \lambda^2) \\ &= 0 \end{aligned}$$

Now this can only be true if either $\alpha_i = 0$, or $i = \lambda$ (remember that $i > 0$ and λ is a positive integer) so that

$$y_N = \cos(\lambda x).$$

The Euler-Lagrange equations

$$\begin{aligned} \frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} &= 0 \\ \frac{d}{dx} 2y' + 2\lambda^2 y &= 0 \\ y'' + \lambda^2 y &= 0 \end{aligned}$$

whose solution we know to be of the form

$$y = A\cos(\lambda x) + B\sin(\lambda x),$$

with $A = 1$ and $B = 0$ given by the end-point conditions.

Obviously, given the trial functions, we can only get a good approximation to this curve when λ is an integer (which was the case here). If λ is not an integer, the solution above suggests the obvious form of trial functions including both sin and cos, resulting in a Fourier series like approximation, which in this problem should give us an exact solution.