Tutorial 6 Solutions

1 Maximize the range of a missile: Take a missile which has a rocket motor that generates constant thrust f for a fixed time interval $[0, t_1]$. We can control the angle of the thrust $\theta(t)$ (relative to the horizontal). Ignoring drag, the curve of the Earth's surface (and its rotation), determine the angle profile that will maximize the range of the missile.

Hints: choose a co-ordinates (x, y), and $(u, v) = (\dot{x}, \dot{y})$, then the DEs describing the system under thrust will be

$$\begin{aligned}
\dot{x} &= u \\
\dot{y} &= v \\
\dot{u} &= f \cos\theta \\
\dot{v} &= f \sin\theta - g
\end{aligned}$$

After the rocket stops firing, the missile will continue on a ballistic trajectory, i.e., the remaining motion will be a parabola, resulting in a total firing distance of

$$R(x, y, u, v) = x + \frac{u}{g}\left[v + \sqrt{v^2 + 2gy}\right]$$

where x, y, u, v are given at the time at which balistic motion commences.

Solutions:

(A). Firstly, note that the problem is of the form maximize

$$F\{\theta\} = R(x, y, u, v) + \int_0^{t_1} 0 \, dt,$$

i.e., the term inside the integral is zero. The only integrand we will get comes from the Lagrange multipliers that describe the system, i.e.,

$$\begin{aligned}
& H\{\theta, x, y, u, v, \lambda_x, \lambda_y, \lambda_u, \lambda_v\} = R(x, y, u, v) \\
+ & \int_0^{t_1} \lambda_x(\dot{x} - u) + \lambda_y(\dot{y} - v) + \lambda_u(\dot{u} - f\cos\theta) + \lambda_v(\dot{v} - f\sin\theta - g)\, dt,
\end{aligned}$$

We get 9 E-L equations, but those with respect to the Lagrange multipliers trivially

give us back the system constraints, so focus on the others, i.e.,

$$\begin{aligned}
\theta: \ \frac{\partial h}{\partial \theta} - \frac{d}{dt}\frac{\partial h}{\partial \dot{\theta}} &= 0 \ \Rightarrow \ \lambda_u f \sin\theta - \lambda_v f \cos\theta &= 0 \\
x: \ \frac{\partial h}{\partial x} - \frac{d}{dt}\frac{\partial h}{\partial \dot{x}} &= 0 \ \Rightarrow \ \dot{\lambda}_x &= 0 \\
y: \ \frac{\partial h}{\partial y} - \frac{d}{dt}\frac{\partial h}{\partial \dot{y}} &= 0 \ \Rightarrow \ \dot{\lambda}_y &= 0 \\
u: \ \frac{\partial h}{\partial u} - \frac{d}{dt}\frac{\partial h}{\partial \dot{u}} &= 0 \ \Rightarrow \ \dot{\lambda}_u &= -\lambda_x \\
v: \ \frac{\partial h}{\partial v} - \frac{d}{dt}\frac{\partial h}{\partial \dot{v}} &= 0 \ \Rightarrow \ \dot{\lambda}_v &= -\lambda_y
\end{aligned}$$

The first equation simplifies to give

$$\tan\theta = \frac{\lambda_v}{\lambda_u}.$$

The next two give

$$\begin{aligned}
\lambda_x &= c_x \\
\lambda_y &= c_y
\end{aligned}$$

where c_x and c_y are constant. The last two equations then give

$$\begin{aligned}
\lambda_u &= -c_x t + c_u \\
\lambda_v &= -c_y t + c_v
\end{aligned}$$

We need to find the constants of integration, so we determine the natural boundary conditions at $t = t_1$. The terminal cost is R, so these take the form (for the non-trivial cases)

$$\begin{aligned}
x: \ \frac{\partial h}{\partial \dot{x}} + \left.\frac{\partial R}{\partial x}\right|_{t=t_1} &= 0 \ \Rightarrow \ \lambda_x(t_1) &= -1 \\
y: \ \frac{\partial h}{\partial \dot{y}} + \left.\frac{\partial R}{\partial y}\right|_{t=t_1} &= 0 \ \Rightarrow \ \lambda_y(t_1) &= \left. -\frac{u}{\sqrt{v^2 + 2gy}}\right|_{t=t_1} \\
u: \ \frac{\partial h}{\partial \dot{u}} + \left.\frac{\partial R}{\partial u}\right|_{t=t_1} &= 0 \ \Rightarrow \ \lambda_u(t_1) &= \left. -\frac{1}{g}\left[v + \sqrt{v^2 + 2gy}\right]\right|_{t=t_1} \\
v: \ \frac{\partial h}{\partial \dot{v}} + \left.\frac{\partial R}{\partial v}\right|_{t=t_1} &= 0 \ \Rightarrow \ \lambda_v(t_1) &= \left. -\frac{u}{g}\left[1 + \frac{v}{\sqrt{v^2 + 2gy}}\right]\right|_{t=t_1}
\end{aligned}$$

The first equation gives $c_x = -1$, and combined with the third this gives

$$\lambda_u(t_1) = t_1 + c_u = -\frac{1}{g}\left[v + \sqrt{v^2 + 2gy}\right]$$

which we can rearrange to give

$$c_u = -t_1 - \frac{1}{g}\left[v + \sqrt{v^2 + 2gy}\right]$$

Likewise

$$\begin{aligned} c_y &= -\frac{u}{\sqrt{v^2+2gy}}\bigg|_{t=t_1} \\ c_v &= -\frac{ut_1}{\sqrt{v^2+2gy}} - \frac{u}{g}\left[1 + \frac{v}{\sqrt{v^2+2gy}}\right]\Bigg|_{t=t_1} \end{aligned}$$

Now

$$\begin{aligned} \tan\theta &= \frac{\lambda_v}{\lambda_u} \\ &= \frac{-c_y t + c_v}{-c_x t + c_u} \\ &= \frac{t - t_1 - \frac{1}{g}\left[v + \sqrt{v^2+2gy}\right]}{t - t_1 - \frac{1}{g}\left[v + \sqrt{v^2+2gy}\right]} \times \frac{u}{\sqrt{v^2+2gy}}\Bigg|_{t=t_1} \\ &= \frac{u}{\sqrt{v^2+2gy}}\bigg|_{t=t_1} \end{aligned}$$

which is a constant! Hence $\theta = const$, to be determined by solving the above equations.

The thrust profile has played no part in the above. It only comes into the solution when we have to calculate θ. Calculating θ simply requires us to substitute constant θ into the system DEs and solve, e.g., for constant thrust f, the system DEs (along with $x(0) = y(0) = u(0) = v(0) = 0$)

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{u} &= f\cos\theta \\ \dot{v} &= f\sin\theta - g \end{aligned}$$

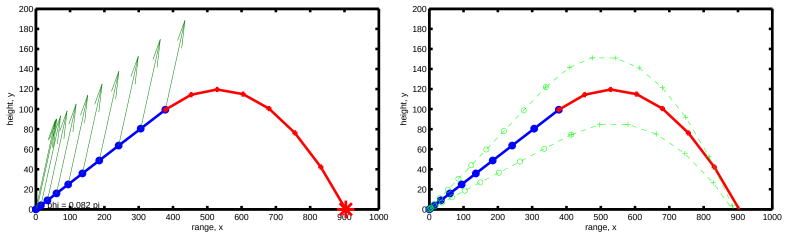
give the behaviour of the rocket under thrust as

$$\begin{aligned} u &= [f\cos\theta]\,t \\ v &= [f\sin\theta - g]\,t \\ x &= [f\cos\theta]\,t^2/2 \\ y &= [f\sin\theta - g]\,t^2/2 \end{aligned}$$

Note that y can only be positive for $f > g$, i.e., the thrust is greater than gravity. From the $\tan\theta = u/\sqrt{v^2+2gy}\big|_{t=t_1}$ which we need to solve for θ, so rearrange thus

$$\begin{aligned} \frac{\sin\theta}{\cos\theta} &= \frac{u(t_1)}{\sqrt{v(t_1)^2 + 2gy(t_1)}} \\ &= \frac{ft_1\cos\theta}{t_1\sqrt{[f\sin\theta - g]^2 + g\,[f\sin\theta - g]}} \\ &= \frac{f\cos\theta}{\sqrt{[f\sin\theta - g]\,[f\sin\theta - g + g]}} \\ &= \frac{\cos\theta}{\sqrt{[\sin\theta - g/f]\sin\theta}} \\ \frac{\sin^2\theta}{\cos^2\theta} &= \frac{\cos^2\theta}{[\sin\theta - g/f]\sin\theta} \\ \sin^3\theta\,[\sin\theta - g/f] &= \cos^4\theta \\ \sin^4\theta - g/f\sin^3\theta &= (1-\sin^2\theta)^2 \\ &= 1 - 2\sin^2\theta + \sin^4\theta \\ -g/f\sin^3\theta &= 1 - 2\sin^2\theta \\ -g/f\sin^3\theta &= \cos 2\theta \\ \sin^3\theta + \frac{f}{g}\cos 2\theta &= 0 \end{aligned}$$

There could be more than one solution to the above, we will choose the one which gives the maximum range. Solve it using `fzero` in Matlab and the following two figures show examples (for $g = 9.8, f = 14$). Dots show position during thrust (arrows show thrust direction) at one second intervals, and $+$ signs show the parabolic arc. The two dashed lines on the right-hand side show the arcs obtained if the angle of thrust is slightly different, and we can see that the best case is our extremal curve.



Note that the trajectory angle is also constant but a different constant from the thrust angle because of gravitational acceleration. We can calculate this angle from $\arctan y/x$, just as we can now calculate all of the quantities including the range R.

2 Conservation laws: Consider the simple 2D harmonic oscillator, i.e, an oscillator whose kinetic and potential energies are described by

$$\begin{aligned} T &= \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) \\ V &= \frac{\omega^2}{2}\left(q_1^2 + q_2^2\right). \end{aligned}$$

(a) Consider whether this system has translation and/or rotational symmetries, and using Noether's theorem describe the conservation laws that apply.

Solutions: The functional of interest is

$$\begin{aligned} J\{\mathbf{q}\} &= \int_{t_0}^{t_1} L(t, \mathbf{q}, \dot{\mathbf{q}}), dt, \\ L &= T - V \\ &= \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2\right) - \frac{\omega^2}{2}\left(q_1^2 + q_2^2\right). \end{aligned}$$

Out of interest, the resulting E-L equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \ddot{q}_i - \omega^2 q_i = 0,$$

and as the solutions to this are simple (independent) sinusoids in each co-ordinate, the reason for naming the system the 2D oscillator should be obvious.

The Lagrangian L is invariant under

- time translations $\Rightarrow$ energy is conserved,
- rotations $\Rightarrow$ angular momentum is conserved.

as shown using Noether's theorem in the notes. The system is not invariant under translations in q_i, and so momentum is not conserved.

(b) Now transform the system using the transform

$$\begin{aligned} x_1 &= \frac{1}{2}(q_1 - iq_2) \\ x_2 &= \frac{1}{2}(q_1 + iq_2). \end{aligned}$$

Show the the resulting system is invariant under the continuous familiy of "squeeze" transforms

$$\begin{aligned} X_1 &= e^{\varepsilon} x_1 \\ X_2 &= e^{-\varepsilon} x_2 \end{aligned}$$

and derive the corresponding conservation law.

Solutions: Inverting the transform we get

$$\begin{aligned} q_1 &= (x_1 + x_2) \\ q_2 &= i(x_1 - x_2). \end{aligned}$$

Applying the transform to the Lagrangian we get

$$L = 2\dot{x}_1\dot{x}_2 - 2\omega^2 x_1 x_2.$$

The obvious point to note is that this Lagrangian is invariant under the transform, because the exponential factors cancel.

This is a relatively little known form for describing the 2D oscillator, but it is interesting to note that the E-L equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 2\ddot{x}_i - 2\omega^2 x_i = 0,$$

which gives the x_i as sinusoids, and which can be transformed back into the original coordinates giving the same solutions.

The transform described, expanded as a Taylor series is

$$\begin{aligned} X_1 &= x_1 + \varepsilon x_1 + \ldots \\ X_2 &= x_2 - \varepsilon x_1 + \ldots \end{aligned}$$

i.e., it has generators

$$\begin{aligned} \eta_1 &= x_1 \\ \eta_2 &= -x_2 \end{aligned}$$

Noether's theorem states that if $L(t, \mathbf{q}, \dot{\mathbf{q}})$ is variationally invariant on $[t_0, t_1]$ under a transform with infinitesimal generators ξ and η_k, then

$$\sum_{k=1}^{n} p_k \eta_k - H\xi = const,$$

along any extremal of

$$F\{\mathbf{x}\} = \int_{t_0}^{t_1} L(t, \mathbf{x}, \dot{\mathbf{x}})\, dt,$$

where

$$p_k = \frac{\partial L}{\partial \dot{x}_k}, \qquad H = \sum_{k=1}^{n} p_k \dot{x}_k - L$$

Here $\xi = 0$, and so the convervation law in question is

$$x_1 p_1 - x_2 p_2 = C, \text{ a constant}.$$

(c) Have we discovered a new conservation law for the system? Explain.

Solutions: On the face of it this might seem like a new conservation law. After all, the motions in the new co-ordinate system are still independent sinusoids, and so the above is not angular momentum, or is it?

Note that from the definition $p_k = \frac{\partial L}{\partial \dot{x}_k}$ so

$$\begin{aligned} p_1 &= 2\dot{x}_2 \\ p_2 &= 2\dot{x}_1, \end{aligned}$$

and so the conserved quantity is actually

$$x_1\dot{x}_2 - x_2\dot{x}_1 = C/2, \text{ a constant}.$$

This looks much more like conventional angular momentum.

Furthermore, if we transform back to the original co-ordinates of the system we have

$$\begin{aligned} x_1 &= \frac{1}{2}(q_1 - iq_2) \\ x_2 &= \frac{1}{2}(q_1 + iq_2) \\ \dot{x}_1 &= \frac{1}{2}(\dot{q}_1 - i\dot{q}_2) \\ \dot{x}_2 &= \frac{1}{2}(\dot{q}_1 + i\dot{q}_2). \end{aligned}$$

and hence

$$\begin{aligned} 4\left[x_1\dot{x}_2 - x_2\dot{x}_1\right] &= (q_1 - iq_2)(\dot{q}_1 + i\dot{q}_2) - (q_1 + iq_2)(\dot{q}_1 - i\dot{q}_2) \\ &= q_1\dot{q}_1 + q_2\dot{q}_2 - iq_2\dot{q}_1 + iq_1\dot{q}_2 - q_1\dot{q}_1 - q_2\dot{q}_2 - iq_2\dot{q}_1 + iq_1\dot{q}_2 \\ &= -2i\left[q_2\dot{q}_1 + q_1\dot{q}_2\right], \end{aligned}$$

which is just $2i\times$ the angular momentum in the original co-ordinates. Hence, the conservation law identified is just conservation of angular momentum.

Note that it is interesting that in an alterantive co-ordinate system, the transform, and resulting symmetry may appear different, but as we might intuitively expect, no new symmetries/conservation laws of the physical system are created by co-ordinate transformation. On the other hand, care must be taken because symmetries that were otherwise unseen may be revealed by a new co-ordinate system, such as the Laplace-Runge-Lenz vector in the orbit of a planet under the inverse square law of gravity.

3. Solve the following optimal control problem: find the control $0 \leq u(t) \leq 1$ that minimizes

$$F\{u\} = \int_0^T x_1 u - x_2 u \, dt$$

subject to the system DEs

$$\begin{aligned} \dot{x}_1 &= 1 - u \\ \dot{x}_2 &= x_1 + 1 \end{aligned}$$

Given starting point $(x_1, x_2) = (0, 0)$ at time 0, and end-point $(x_1, x_2) = (1, 2)$ derive the time T at which we reach the end-point.

Solution:

The Hamiltonian is

$$H = -x_1 u + x_2 u + p_1(1 - u) + p_2(x_1 + 1).$$

The Hamiltonian is clearly linear in u, and so the control will be a bang-bang controller, with switching function

$$\sigma = -x_1 + x_2 - p_1.$$

So the control will be

$$u = \begin{cases} 1 & \text{if } -x_1 + x_2 - p_1 > 0 \\ 0 & \text{if } -x_1 + x_2 - p_1 < 0 \end{cases}$$

Considering the two possible cases we can solve the system DEs to get

(a) $u = 1$ the system DEs are

$$\begin{aligned} \dot{x}_1 &= 0 \\ \dot{x}_2 &= x_1 + 1 \end{aligned}$$

so clearly the solution is

$$\begin{aligned} x_1 &= c_1 \\ x_2 &= (c_1 + 1)t + c_2 \end{aligned}$$

So x_1 is constant, and hence in the phase space, the paths are vertical lines with arrows in the upwards direction.

(b) $u = 0$ the system DEs are

$$\begin{aligned} \dot{x}_1 &= 1 \\ \dot{x}_2 &= x_1 + 1 \end{aligned}$$

so clearly the solution is

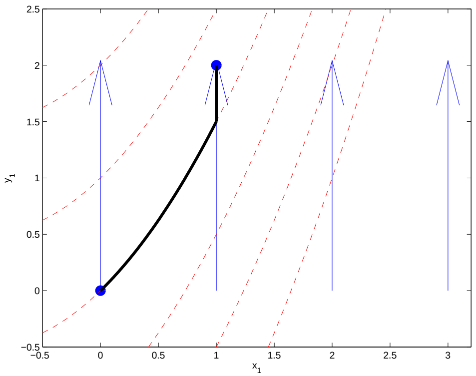
$$\begin{aligned} x_1 &= t + c_3 \\ x_2 &= t^2/2 + (c_3+1)t + c_4 \end{aligned}$$

Writing x_2 as a function of x_1 we get

$$x_2(x_1) = \frac{x_1^2}{2} + x_1 + c_5$$

where $c_3 = c_4 - c_3^2 - c_3$. So these are quadratic curves in the phase space, where x_2 increases with x_1.

The problem is linear and autonomous, and the process has $n = 2$, so it can have at most one switch point. The phase diagram is shown below (solid arrow show $u = 1$ and dashed lines $u = 0$) from which we can see the a possible path for the given end-points (shown as the solid line).



Note that regardless of the starting and finishing point, once we know that $u = 0$ or 1, and the form of x_1, x_2 in the later case, we can easily calculate the integral, i.e., if the first phase is $u = 0$ followed by $u = 1$ the integral is

$$\begin{aligned} F\{u\} &= \int_0^T x_1 u - x_2 u \; dt \\ &= \int_{t_s}^T x_1 - x_2 \; dt \\ &= \int_{t_s}^T c_1 - ((c_1+1)t + c_2) \; dt \\ &= \left[(c_1 - c_2)t - (c_1+1)t^2/2\right]_{t_s}^T . \end{aligned}$$

Likewise, if the first phase is $u = 1$ followed by $u = 0$ then

$$\begin{aligned} F\{u\} &= \left[(c_1 - c_2)t - (c_1+1)t^2/2\right]_0^{t_s} \\ &= (c_1 - c_2)t_s - (c_1+1)t_s^2/2. \end{aligned}$$

In the path in the figure we see we start in the phase $u = 0$, and then switch to the phase $u = 1$. It is easy to derive (from the initial point at $t = 0$) that $c_3 = c_4 = 0$.

The switch point must occur at $x_1(t_s) = 1$, so , we can see that it occurs at $t_s = 1$, at which point $x_2 = 1.5$.

We can then derive the constants for the second phase of motion $c_1 = 1$, and $c_2 = -0.5$. From this, we can determine that we will reach the end-point at time T such that $x_2(T) = 2T - 1/2 = 2$, namely $T = 5/4$.

To assess the other possible paths we would go through the same process and calculate the integral again to see which is better (in this case this one is best).

The above omitts consideration of potential singular controls, which we could assess by solving the canonical EL equations to obtain the conjugate momentum terms, and checking that $\sigma \neq 0$.