THE DETERMINANT OF A TRIANGULAR-BLOCK MATRIX

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1. Problem. A useful way of representing a system that is a multiple-input, multiple-output digital filter uses a matrix to describe the linear transformation of the concatenated inputs to the concatenated outputs. This matrix can be partitioned into lower triangular matrices that have the additional property of being Toeplitz. It is useful to be able to efficiently calculate the determinant and eigenvalues of such a matrix.

(i) Let M be a square matrix of size $KL \times KL$. Let M be partitioned into K^2 lower-triangular sub-matrices of size $L \times L$, denoted $S^{(ij)}$. Thus

$$M = \begin{bmatrix} S^{(11)} & S^{(12)} & \dots & S^{(1K)} \\ S^{(21)} & S^{(22)} & \dots & S^{(2K)} \\ \vdots & \vdots & \ddots & \vdots \\ S^{(K1)} & S^{(K2)} & \dots & S^{(KK)} \end{bmatrix}.$$

where

$$S^{(ij)} = \begin{bmatrix} S^{(ij)}_{11} & 0 & \cdots & 0 \\ S^{(ij)}_{21} & S^{(ij)}_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ S^{(ij)}_{L1} & \cdots & S^{(ij)}_{L(L-1)} & S^{(ij)}_{LL} \end{bmatrix}.$$

Find a form of the determinant of M which can be efficiently computed for large L.

(ii) Hence or otherwise show that when the sub-matrices $S^{(ij)}$ are also Toeplitz, *i.e.*

$$S^{(ij)} = \begin{bmatrix} a^{(ij)} & 0 & \cdots & 0 \\ b^{(ij)} & a^{(ij)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ z^{(ij)} & \dots & b^{(ij)} & a^{(ij)} \end{bmatrix},$$

the determinant is given by $\det(M) = \det(N)^L$, where N is the $K \times K$ matrix whose elements are the main diagonal elements of the sub-matrices S_{ij}, *i.e.*

$$N = \begin{bmatrix} a^{(11)} & a^{(12)} & \dots & a^{(1K)} \\ a^{(21)} & a^{(22)} & \dots & a^{(2K)} \\ \vdots & \vdots & \ddots & \vdots \\ a^{(K1)} & a^{(K2)} & \dots & a^{(KK)} \end{bmatrix}.$$

(iii) Show that the eigenvalues of M are independent of the off-diagonal elements of the sub-matrices $S^{(ij)}$ and that if the sub-matrices are Toeplitz then the eigenvalues of M are the same as those of N defined above.

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(iv) Can the result in **(i)** be extended to the case where some of the sub-matrices are upper triangular while others are lower triangular or to the case when the sub-matrices are Hessenburg?

2. Solutions. (i) The solution is based on a reordering of rows and columns. The matrix M is rearranged to give the matrix $\tilde{M}$, whose first K columns equal columns $1, L+1, 2L+1, \ldots, (K-1)L+1$ of M. The second K columns equal columns $2, L+2, 2L+2, \ldots, (K-1)L+2$ of M, and similarly for the remaining columns of $\tilde{M}$. An identical rearrangement of the rows is then also effected.

The reordering can be written as the post- and pre-multiplication of M by a permutation matrix P and its transpose respectively. P can written as a matrix partitioned into $K \times L$ rectangular matrices of size $L \times K$, eg:

$$P = \begin{bmatrix} R^{(11)} & R^{(12)} & \cdots & R^{(1L)} \\ R^{(21)} & R^{(22)} & \cdots & R^{(2L)} \\ \vdots & \vdots & \ddots & \vdots \\ R^{(K1)} & R^{(K2)} & \cdots & R^{(KL)} \end{bmatrix},$$

where the (i,j)th sub-matrix has only one non-zero element, being the (j,i)th element. Thus the $R^{(ij)}$ are defined by

$$R^{(ij)}_{kl} = \begin{cases} 1 & \text{if k=j and l=i} \\ 0 & \text{otherwise} \end{cases}.$$

Thus the matrix $\tilde{M}$ is given by

$$\tilde{M} = P^{\mathrm{T}} M P,$$

and furthermore, $\tilde{M}$ can be partitioned into L^2 sub-matrices of size $K \times K$ as follows:

$$\tilde{M} = \begin{bmatrix} N^{(11)} & 0 & \cdots & 0 \\ N^{(21)} & N^{(22)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ N^{(L1)} & N^{(L2)} & \cdots & N^{(LL)} \end{bmatrix},$$

where $N^{(ij)}$ is defined by

$$N^{(ij)} = \begin{bmatrix} S^{(11)}_{ij} & S^{(12)}_{ij} & \cdots & S^{(1K)}_{ij} \\ S^{(21)}_{ij} & S^{(22)}_{ij} & \cdots & S^{(2K)}_{ij} \\ \vdots & \vdots & \ddots & \vdots \\ S^{(K1)}_{ij} & S^{(K2)}_{ij} & \cdots & S^{(KK)}_{ij} \end{bmatrix}.$$

Thus $\tilde{M}$ is block triangular, and hence its determinant is the product of the determinants of the main diagonal blocks [1], *i.e.*

$$\det(\tilde{M}) = \prod_{i=1}^{L} \det\left(N^{(ii)}\right).$$

Note $P^{\mathrm{T}} = P^{-1}$ and so

$$\begin{aligned} \det(M) &= \det(P^{\mathrm{T}} \tilde{M} P) \\ &= \det(P)^{-1} \det(\tilde{M}) \det(P) \\ &= \prod_{i=1}^{L} \det\left(N^{(ii)}\right), \end{aligned} \tag{1}$$

which requires a number of operations proportional to L to calculate.

(ii) The problem is a special case of **(i)** in which

$$N^{(ii)} = N = \begin{bmatrix} a^{(11)} & a^{(12)} & \cdots & a^{(1K)} \\ a^{(21)} & a^{(22)} & \cdots & a^{(2K)} \\ \vdots & \vdots & \ddots & \vdots \\ a^{(K1)} & a^{(K2)} & \cdots & a^{(KK)} \end{bmatrix},$$

and so from Equation (1) the determinant can be written $\det(M) = \det(N)^L$.

(iii) The matrix $M - \lambda I$ has the same block structure as M. From Equation (1)

$$\det(M - \lambda I) = \prod_{i=1}^{L} \det\left(N^{(ii)} - \lambda I\right),$$

and so the eigenvalues of M are the same as those of $N^{(11)} \oplus \cdots \oplus N^{(LL)}$ and the $N^{(ii)}$ depend only on the diagonal elements of the sub-matrices $S^{(ij)}$. When the sub-matrices are Toeplitz, $N^{(ii)} = N$.

(iv) Unsolved.

REFERENCES

[1] R. A. Horn and C. R. Johnson. *Matrix Analysis.* Cambridge University Press, 1990.