An Algebraic Approach to Internet Routing
Day 1

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Semigroups

Definition (Semigroup)

A semigroup $(S, \oplus)$ is a non-empty set S with a binary operation such that

$$\text{ASSOCIATIVE} \quad : \quad a \oplus (b \oplus c) = (a \oplus b) \oplus c$$

S	$\oplus$	where
$\mathbb{N}^{\infty}$	min	
$\mathbb{N}^{\infty}$	max	
$\mathbb{N}^{\infty}$	$+$	
2^W	$\cup$	
2^W	$\cap$	
S^*	$\circ$	$(abc \circ de = abcde)$
S	left	$(a \text{ left } b = a)$
S	right	$(a \text{ right } b = b)$

Special Elements

Definition

- $\alpha \in S$ is an identity if for all $a \in S$

$$a = \alpha \oplus a = a \oplus \alpha$$

- A semigroup is a monoid if it has an identity.
- ω is an annihilator if for all $a \in S$

$$\omega = \omega \oplus a = a \oplus \omega$$

S	$\oplus$	α	ω
$\mathbb{N}^\infty$	min	∞	0
$\mathbb{N}^\infty$	max	0	∞
$\mathbb{N}^\infty$	$+$	0	∞
2^W	$\cup$	$\{\}$	W
2^W	$\cap$	W	$\{\}$
S^*	$\circ$	ϵ	
S	left		
S	right		

Important Properties

Definition (Some Important Semigroup Properties)

$$\begin{array}{rcrcl} \text{COMMUTATIVE} & : & a \oplus b & = & b \oplus a \\ \text{SELECTIVE} & : & a \oplus b & \in & \{a, b\} \\ \text{IDEMPOTENT} & : & a \oplus a & = & a \end{array}$$

S	$\oplus$	COMMUTATIVE	SELECTIVE	IDEMPOTENT
$\mathbb{N}^\infty$	min	$\star$	$\star$	$\star$
$\mathbb{N}^\infty$	max	$\star$	$\star$	$\star$
$\mathbb{N}^\infty$	$+$	$\star$		
2^W	$\cup$	$\star$		$\star$
2^W	$\cap$	$\star$		$\star$
S^*	$\circ$			
S	left		$\star$	$\star$
S	right		$\star$	$\star$

Order Relations

We are interested in order relations $\leq \subseteq S \times S$

Definition (Important Order Properties)

$$
\begin{array}{rcl}
\text{REFLEXIVE} & : & a \leq a \\
\text{TRANSITIVE} & : & a \leq b \wedge b \leq c \rightarrow a \leq c \\
\text{ANTISYMMETRIC} & : & a \leq b \wedge b \leq a \rightarrow a = b \\
\text{TOTAL} & : & a \leq b \vee b \leq a
\end{array}
$$

	pre-order	partial order	preference order	total order
REFLEXIVE	⋆	⋆	⋆	⋆
TRANSITIVE	⋆	⋆	⋆	⋆
ANTISYMMETRIC		⋆		⋆
TOTAL			⋆	⋆

Canonical Pre-order of a Commutative Semigroup

Suppose $\oplus$ is commutative.

Definition (Canonical pre-orders)

$$a \trianglelefteq_{\oplus}^{R} b \equiv \exists c \in S : b = a \oplus c$$
$$a \trianglelefteq_{\oplus}^{L} b \equiv \exists c \in S : a = b \oplus c$$

Lemma (Sanity check)

Associativity of $\oplus$ implies that these relations are transitive.

Proof.

Note that $a \trianglelefteq_{\oplus}^{R} b$ means $\exists c_1 \in S : b = a \oplus c_1$, and $b \trianglelefteq_{\oplus}^{R} c$ means $\exists c_2 \in S : c = b \oplus c_2$. Letting $c_3 = c_1 \oplus c_2$ we have $c = b \oplus c_2 = (a \oplus c_1) \oplus c_2 = a \oplus (c_1 \oplus c_2) = a \oplus c_3$. That is, $\exists c_3 \in S : c = a \oplus c_3$, so $a \trianglelefteq_{\oplus}^{R} c$. The proof for $\trianglelefteq_{\oplus}^{L}$ is similar. □

Canonically Ordered Semigroup

Definition (Canonically Ordered Semigroup)

A commutative semigroup $(S, \oplus)$ is canonically ordered when $a \trianglelefteq^R_\oplus c$ and $a \trianglelefteq^L_\oplus c$ are partial orders.

Definition (Groups)

A monoid is a group if for every $a \in S$ there exists a $a^{-1} \in S$ such that $a \oplus a^{-1} = a^{-1} \oplus a = \alpha$.

Canonically Ordered Semigroups vs. Groups [Car79, GM08]

Lemma (THE BIG DIVIDE)

Only a trivial group is canonically ordered.

Proof.

If $a, \ b \in S$, then $a = \alpha_{\oplus} \oplus a = (b \oplus b^{-1}) \oplus a = b \oplus (b^{-1} \oplus a) = b \oplus c$, for $c = b^{-1} \oplus a$, so $a \trianglelefteq^{L}_{\oplus} b$. In a similar way, $b \trianglelefteq^{R}_{\oplus} a$. Therefore $a = b$. □

Natural Orders

Definition (Natural orders)

Let $(S, \oplus)$ be a semigroup.

$$
\begin{aligned}
a \leq^{L}_{\oplus} b & \equiv & a = a \oplus b \\
a \leq^{R}_{\oplus} b & \equiv & b = a \oplus b
\end{aligned}
$$

Lemma

If $\oplus$ is commutative and idempotent, then $a \trianglelefteq^{D}_{\oplus} b \iff a \leq^{D}_{\oplus} b$, for $D \in \{R, L\}$.

Proof.

$$
\begin{aligned}
a \trianglelefteq^{R}_{\oplus} b & \iff & b = a \oplus c = (a \oplus a) \oplus c = a \oplus (a \oplus c) \\
& = & a \oplus b \iff a \leq^{R}_{\oplus} b \\
a \trianglelefteq^{L}_{\oplus} b & \iff & a = b \oplus c = (b \oplus b) \oplus c = b \oplus (b \oplus c) \\
& = & b \oplus a = a \oplus b \iff a \leq^{L}_{\oplus} b
\end{aligned}
$$

□

Special elements and natural orders

Lemma (Natural Bounds)

- *If α exists, then for all a, $a \leq_{\oplus}^{L} \alpha$ and $\alpha \leq_{\oplus}^{R} a$*
- *If ω exists, then for all a, $\omega \leq_{\oplus}^{L} a$ and $a \leq_{\oplus}^{R} \omega$*
- *If α and ω exist, then S is bounded.*

$$\begin{array}{ccccc} \omega & \leq_{\oplus}^{L} & a & \leq_{\oplus}^{L} & \alpha \\ \alpha & \leq_{\oplus}^{R} & a & \leq_{\oplus}^{R} & \omega \end{array}$$

Remark (Thanks to Iljitsch van Beijnum)

Note that this means for (min, $+$) we have

$$\begin{array}{ccccc} 0 & \leq_{\min}^{L} & a & \leq_{\min}^{L} & \infty \\ \infty & \leq_{\min}^{R} & a & \leq_{\min}^{R} & 0 \end{array}$$

and still say that this is bounded, even though one might argue with the terminology!

Examples of special elements

S	$\oplus$	α	ω	$\leq^{L}_{\oplus}$	$\leq^{R}_{\oplus}$
$\mathbb{N} \cup \{\infty\}$	min	∞	0	$\leq$	$\geq$
$\mathbb{N} \cup \{\infty\}$	max	0	∞	$\geq$	$\leq$
$\mathcal{P}(W)$	$\cup$	$\{\}$	W	$\supseteq$	$\subseteq$
$\mathcal{P}(W)$	$\cap$	W	$\{\}$	$\subseteq$	$\supseteq$

Property Management

Lemma

Let $D \in \{R,\ L\}$.

1. $\text{IDEMPOTENT}((S,\ \oplus)) \iff \text{REFLEXIVE}((S,\ \leq_{\oplus}^{D}))$
2. $\text{COMMUTATIVE}((S,\ \oplus)) \implies \text{ANTISYMMETRIC}((S,\ \leq_{\oplus}^{D}))$
3. $\text{SELECTIVE}((S,\ \oplus)) \iff \text{TOTAL}((S,\ \leq_{\oplus}^{D}))$

Proof.

1. $a \leq_{\oplus}^{D} a \iff a = a \oplus a,$
2. $a \leq_{\oplus}^{L} b \wedge b \leq_{\oplus}^{L} a \iff a = a \oplus b \wedge b = b \oplus a \implies a = b$
3. $a = a \oplus b \vee b = a \oplus b \iff a \leq_{\oplus}^{L} b \vee b \leq_{\oplus}^{L} a$

□

Direct Product of Semigroups

Let $(S, \oplus_S)$ and $(T, \oplus_T)$ be semigroups.

Definition (Direct product semigroup)

The direct product is denoted $(S, \oplus_S) \times (T, \oplus_T) = (S \times T, \oplus)$, where $\oplus = \oplus_S \times \oplus_T$ is defined as

$$(s_1, t_1) \oplus (s_2, t_2) = (s_1 \oplus_S s_2,\ t_1 \oplus_T t_2).$$

Lexicographic Product of Semigroups

Definition (Lexicographic product semigroup (from [Gur08]))

Suppose S is commutative idempotent semigroup and T be a monoid. The lexicographic product is denoted $(S, \oplus_S) \vec{\times} (T, \oplus_T) = (S \times T, \oplus)$, where $\vec{\oplus} = \oplus_S \vec{\times} \oplus_T$ is defined as

$$(s_1, t_1) \vec{\oplus} (s_2, t_2) = \begin{cases} (s_1 \oplus_S s_2, t_1 \oplus_T t_2) & s_1 = s_1 \oplus_S s_2 = s_2 \\ (s_1 \oplus_S s_2, t_1) & s_1 = s_1 \oplus_S s_2 \neq s_2 \\ (s_1 \oplus_S s_2, t_2) & s_1 \neq s_1 \oplus_S s_2 = s_2 \\ (s_1 \oplus_S s_2, \overline{0}_T) & \text{otherwise.} \end{cases}$$

Semirings

$(S, \oplus, \otimes, \overline{0}, \overline{1})$ is a semiring when

- $(S, \oplus, \overline{0})$ is a commutative monoid
- $(S, \otimes, \overline{1})$ is a monoid
- $\overline{0}$ is an annihilator for $\otimes$

and distributivity holds,

$$
\begin{array}{lrcl}
\mathrm{LD} & : \; a \otimes (b \oplus c) & = & (a \otimes b) \oplus (a \otimes c) \\
\mathrm{RD} & : \; (a \oplus b) \otimes c & = & (a \otimes c) \oplus (b \otimes c)
\end{array}
$$

A few examples

name	S	$\oplus$,	$\otimes$	$\overline{0}$	$\overline{1}$	possible routing use
sp	$\mathbb{N}^{\infty}$	min	$+$	∞	0	minimum-weight routing
bw	$\mathbb{N}^{\infty}$	max	min	0	∞	greatest-capacity routing
rel	$[0, 1]$	max	$\times$	0	1	most-reliable routing
use	$\{0, 1\}$	max	min	0	1	usable-path routing
	2^W	$\cup$	$\cap$	$\{\}$	W	shared link attributes?
	2^W	$\cap$	$\cup$	W	$\{\}$	shared path attributes?

Encoding path problems

- $(S,\ \oplus,\ \otimes,\ \overline{0},\ \overline{1})$ a semiring
- $G = (V,\ E)$ a directed graph
- $w \in E \rightarrow S$ a weight function

Path weight

The *weight* of a path $p = i_1, i_2, i_3, \cdots, i_k$ is

$$w(p) = w(i_1,\ i_2) \otimes w(i_2,\ i_3) \otimes \cdots \otimes w(i_{k-1},\ i_k).$$

The empty path is given the weight $\overline{1}$.

Adjacency matrix **A**

$$\mathbf{A}(i,\ j) = \begin{cases} w(i,\ j) & \text{if } (i,\ j) \in E, \\ \overline{0} & \text{otherwise} \end{cases}$$

The general problem of finding globally optimal paths

Given an adjacency matrix **A**, find **R** such that for all $i,\ j \in V$

$$\mathbf{R}(i,\ j) = \bigoplus_{p \in P(i,\ j)} w(p)$$

How can we solve this problem?

Powers and closure

- $(S, \oplus, \otimes, \overline{0}, \overline{1})$ a semiring

Powers, a^k

$$\begin{aligned} a^0 &= \overline{1} \\ a^{k+1} &= a \otimes a^k \end{aligned}$$

Closure, a^*

$$\begin{aligned} a^{(k)} &= a^0 \oplus a^1 \oplus a^2 \oplus \cdots \oplus a^k \\ a^* &= a^0 \oplus a^1 \oplus a^2 \oplus \cdots \oplus a^k \oplus \cdots \end{aligned}$$

Fun Facts [Con71]

$$\begin{aligned} (a^*)^* &= a^* \\ (a \oplus b)^* &= (a^* b)^* a^* \\ (ab)^* &= \overline{1} \oplus a(ba)^* b \end{aligned}$$

Stability

Definition (q stability)

If there exists a q such that $a^{(q)} = a^{(q+1)}$, then a is q-stable. Therefore, $a^* = a^{(q)}$, assuming $\oplus$ is idempotent.

Fact 1

If $\overline{1}$ is an annihiltor for $\oplus$, then every $a \in S$ is 0-stable!

Lift semiring to matrices

- $(S, \oplus, \otimes, \overline{0}, \overline{1})$ a semiring
- Define the semiring of $n \times n$-matrices over S : $(\mathbb{M}_n(S), \oplus, \otimes, \mathbf{J}, \mathbf{I})$

$\oplus$ and $\otimes$

$$\begin{aligned}(\mathbf{A} \oplus \mathbf{B})(i, j) &= \mathbf{A}(i, j) \oplus \mathbf{B}(i, j) \\ (\mathbf{A} \otimes \mathbf{B})(i, j) &= \bigoplus_{1 \le q \le n} \mathbf{A}(i, q) \otimes \mathbf{B}(q, j)\end{aligned}$$

J and I

$$\begin{aligned}\mathbf{J}(i, j) &= \overline{0} \\ \mathbf{I}(i, j) &= \begin{cases} \overline{1} & (\text{if } i = j) \\ \overline{0} & (\text{otherwise}) \end{cases}\end{aligned}$$

$\mathbb{M}_n(S)$ is a semiring!

Check (left) distribution

$$\mathbf{A} \otimes (\mathbf{B} \oplus \mathbf{C}) = (\mathbf{A} \otimes \mathbf{B}) \oplus (\mathbf{A} \otimes \mathbf{C})$$

$$\begin{aligned}
& (\mathbf{A} \otimes (\mathbf{B} \oplus \mathbf{C}))(i,\ j) \\
= & \bigoplus_{1 \le q \le n} \mathbf{A}(i,\ q) \otimes (\mathbf{B} \oplus \mathbf{C})(q,\ j) \\
= & \bigoplus_{1 \le q \le n} \mathbf{A}(i,\ q) \otimes (\mathbf{B}(q,\ j) \oplus \mathbf{C}(q,\ j)) \\
= & \bigoplus_{1 \le q \le n} (\mathbf{A}(i,\ q) \otimes \mathbf{B}(q,\ j)) \oplus (\mathbf{A}(i,\ q) \otimes \mathbf{C}(q,\ j)) \\
= & (\bigoplus_{1 \le q \le n} \mathbf{A}(i,\ q) \otimes \mathbf{B}(q,\ j)) \oplus (\bigoplus_{1 \le q \le n} \mathbf{A}(i,\ q) \otimes \mathbf{C}(q,\ j)) \\
= & ((\mathbf{A} \otimes \mathbf{B}) \oplus (\mathbf{A} \otimes \mathbf{C}))(i,\ j)
\end{aligned}$$

On the matrix semiring

Matrix powers, $\mathbf{A}^k$

$$\mathbf{A}^0 = \mathbf{I}$$

$$\mathbf{A}^{k+1} = \mathbf{A} \otimes \mathbf{A}^k$$

Closure, $\mathbf{A}^*$

$$\mathbf{A}^{(k)} = \mathbf{I} \oplus \mathbf{A}^1 \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^k$$

$$\mathbf{A}^* = \mathbf{I} \oplus \mathbf{A}^1 \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^k \oplus \cdots$$

Note: $\mathbf{A}^*$ might not exist (sum may not converge)

Fact 2

If S is 0-stable, then $\mathbb{M}_n(S)$ is $(n-1)$-stable. That is,

$$\mathbf{A}^* = \mathbf{A}^{(n-1)} = \mathbf{I} \oplus \mathbf{A}^1 \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^{n-1}$$

Computing optimal paths

- Let $P(i,j)$ be the set of paths from i to j.
- Let $P^k(i,j)$ be the set of paths from i to j with exactly k arcs.
- Let $P^{(k)}(i,j)$ be the set of paths from i to j with at most k arcs.

Theorem

$$\begin{array}{lrcl} (1) & \mathbf{A}^k(i,\ j) & = & \bigoplus_{p \in P^k(i,\ j)} w(p) \\ (2) & \mathbf{A}^{(k+1)}(i,\ j) & = & \bigoplus_{p \in P^{(k)}(i,\ j)} w(p) \\ (3) & \mathbf{A}^*(i,\ j) & = & \bigoplus_{p \in P(i,\ j)} w(p) \end{array}$$

Proof of (1)

By induction on k. Base Case: $k = 0$.

$$P^0(i,\ i) = \{\epsilon\},$$

so $\mathbf{A}^0(i\ , i) = \mathbf{I}(i,\ i) = \overline{1} = w(\epsilon)$.

And $i \neq j$ implies $P^0(i,j) = \{\}$. By convention

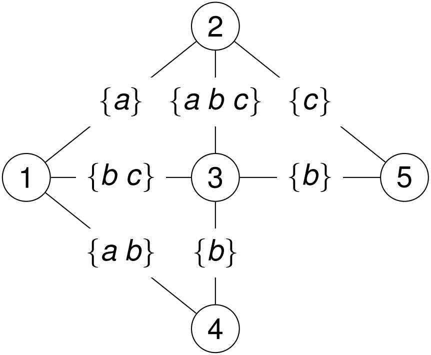
$$\bigoplus_{p \in \{\}} w(p) = \overline{0} = \mathbf{I}(i,\ j).$$

Proof of (1)

Induction step.

$$
\begin{aligned}
\mathbf{A}^{k+1}(i,j) &= (\mathbf{A} \otimes \mathbf{A}^k)(i,\ j) \\
&= \bigoplus_{1 \le q \le n} \mathbf{A}(i,\ q) \otimes \mathbf{A}^k(q,\ j) \\
&= \bigoplus_{1 \le q \le n} \mathbf{A}(i,\ q) \otimes (\bigoplus_{p \in P^k(q,\ j)} w(p)) \\
&= \bigoplus_{1 \le q \le n} \bigoplus_{p \in P^k(q,\ j)} \mathbf{A}(i,\ q) \otimes w(p) \\
&= \bigoplus_{(i,\ q) \in E} \bigoplus_{p \in P^k(q,j)} w(i,\ q) \otimes w(p) \\
&= \bigoplus_{p \in P^{k+1}(i,\ j)} w(p)
\end{aligned}
$$

Example with $(2^{\{a, b, c\}}, \cap, \cup)$



We want matrix $\mathbf{A}^*$ to solve this global optimality problem:

$$\mathbf{A}^*(i, j) = \bigcap_{p \in P(i, j)} w(p),$$

where $w(p)$ is now the union of all edge weights in p.

For $x \in \{a, b, c\}$, interpret $x \in \mathbf{A}^*(i, j)$ to mean that every path from i to j has at least one arc with weight containing x.

$(2^{\{a,\ b,\ c\}},\ \cap,\ \cup)$ continued ...

The matrix $\mathbf{A}^*$

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\left[\begin{array}{ccccc}
\{\} & \{\} & \{b\} & \{b\} & \{\} \\
\{\} & \{\} & \{b\} & \{b\} & \{\} \\
\{b\} & \{b\} & \{\} & \{b\} & \{b\} \\
\{b\} & \{b\} & \{b\} & \{\} & \{b\} \\
\{\} & \{\} & \{b\} & \{b\} & \{\}
\end{array}\right]
\end{array}
$$

Partition Equation (left)

$$\mathbf{X} = (\mathbf{A}\mathbf{X}) \oplus \mathbf{I}$$

$$\left(\begin{array}{c|c} \mathbf{X}_{1,1} & \mathbf{X}_{1,2} \\ \hline \mathbf{X}_{2,1} & \mathbf{X}_{2,2} \end{array}\right)$$

$$= \left(\begin{array}{c|c} (\mathbf{A}_{1,1}\mathbf{X}_{1,1}) \oplus (\mathbf{A}_{1,2}\mathbf{X}_{2,1}) \oplus \mathbf{I}_{1,1} & (\mathbf{A}_{1,1}\mathbf{X}_{1,2}) \oplus (\mathbf{A}_{1,2}\mathbf{X}_{2,2}) \\ \hline (\mathbf{A}_{2,1}\mathbf{X}_{1,1}) \oplus (\mathbf{A}_{2,2}\mathbf{X}_{2,1}) & (\mathbf{A}_{2,1}\mathbf{X}_{1,2}) \oplus (\mathbf{A}_{2,2}\mathbf{X}_{2,2}) \oplus \mathbf{I}_{2,2} \end{array}\right)$$

We now have four (left) equations

$$
\begin{aligned}
\mathbf{X}_{1,1} &= (\mathbf{A}_{1,1}\mathbf{X}_{1,1}) \oplus (\mathbf{A}_{1,2}\mathbf{X}_{2,1}) \oplus \mathbf{I}_{1,1} \\
\mathbf{X}_{2,1} &= (\mathbf{A}_{2,1}\mathbf{X}_{1,1}) \oplus (\mathbf{A}_{2,2}\mathbf{X}_{2,1}) \\
\mathbf{X}_{1,2} &= (\mathbf{A}_{1,1}\mathbf{X}_{1,2}) \oplus (\mathbf{A}_{1,2}\mathbf{X}_{2,2}) \\
\mathbf{X}_{2,2} &= (\mathbf{A}_{2,1}\mathbf{X}_{1,2}) \oplus (\mathbf{A}_{2,2}\mathbf{X}_{2,2}) \oplus \mathbf{I}_{2,2}
\end{aligned}
$$

- Solve for $\mathbf{X}_{2,1}$ with $\mathbf{A}^*_{2,2}\mathbf{A}_{2,1}\mathbf{X}_{1,1}$
- Therefore

$$
\begin{aligned}
& \mathbf{X}_{1,1} \\
= \; & (\mathbf{A}_{1,1}\mathbf{X}_{1,1}) \oplus (\mathbf{A}_{1,2}\mathbf{A}^*_{2,2}\mathbf{A}_{2,1}\mathbf{X}_{1,1}) \oplus \mathbf{I}_{1,1} \\
= \; & (\mathbf{A}_{1,1} \oplus \mathbf{A}_{1,2}\mathbf{A}^*_{2,2}\mathbf{A}_{2,1})\mathbf{X}_{1,1} \oplus \mathbf{I}_{1,1}
\end{aligned}
$$

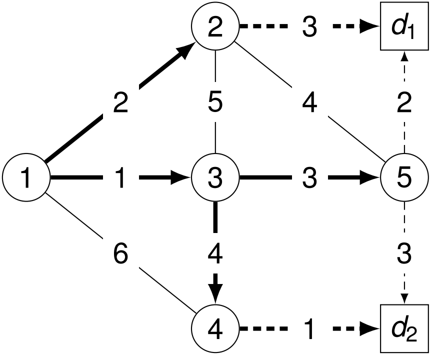
- Solve for $\mathbf{X}_{1,1}$ with $(\mathbf{A}_{1,1} \oplus \mathbf{A}_{1,2}\mathbf{A}^*_{2,2}\mathbf{A}_{2,1})^*$
- So $\mathbf{X}_{2,1}$ is solved with $\mathbf{A}^*_{2,2}\mathbf{A}_{2,1}(\mathbf{A}_{1,1} \oplus \mathbf{A}_{1,2}\mathbf{A}^*_{2,2}\mathbf{A}_{2,1})^*$
- In a similar way, solve for $\mathbf{X}_{1,2}$ and $\mathbf{X}_{2,2}$

This gives a partition of $\mathbf{A}^*$ [Con71]

$\mathbf{A}^*$

$$\left(\begin{array}{c|c} (\mathbf{A}_{1,1} \oplus \mathbf{A}_{1,2}\mathbf{A}_{2,2}^*\mathbf{A}_{2,1})^* & \mathbf{A}_{1,1}^*\mathbf{A}_{1,2}(\mathbf{A}_{2,2} \oplus \mathbf{A}_{2,1}\mathbf{A}_{1,1}^*\mathbf{A}_{1,2})^* \\ \hline \mathbf{A}_{2,2}^*\mathbf{A}_{2,1}(\mathbf{A}_{1,1} \oplus \mathbf{A}_{1,2}\mathbf{A}_{2,2}^*\mathbf{A}_{2,1})^* & (\mathbf{A}_{2,2} \oplus \mathbf{A}_{2,1}\mathbf{A}_{1,1}^*\mathbf{A}_{1,2})^* \end{array}\right)$$

Trivial example of forwarding = routing + mapping



$$\mathbf{M} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} \infty & \infty \\ 3 & \infty \\ \infty & \infty \\ \infty & 1 \\ 2 & 3 \end{bmatrix} \end{array}$$

Mapping matrix

matrix	solves
$\mathbf{A}^*$	$\mathbf{R} = (\mathbf{A} \otimes \mathbf{R}) \oplus \mathbf{I}$
$\mathbf{A}^*\mathbf{M}$	$\mathbf{F} = (\mathbf{A} \otimes \mathbf{F}) \oplus \mathbf{M}$

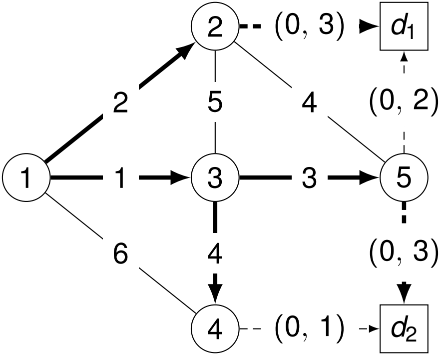
$$\mathbf{F} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} 5 & 6 \\ 3 & 7 \\ 5 & 5 \\ 9 & 1 \\ 2 & 3 \end{bmatrix} \end{array}$$

Forwarding matrix

Routing Matrix vs. Forwarding Matrix (see [BG09])

- Inspired by the the Locator/ID split work
  - See Locator/ID Separation Protocol (LISP)
- Let's make a distinction between infrastructure nodes V and destinations D.
- Assume $V \cap D = \{\}$
- $\mathbf{M}$ is a $V \times D$ mapping matrix
  - $\mathbf{M}(v, d) \neq \infty$ means that destination (identifier) d is somehow attached to node (locator) v

More Interesting Example : Hot-Potato Idiom



$$\mathbf{M} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} \infty & \infty \\ (0, 3) & \infty \\ \infty & \infty \\ \infty & (0, 1) \\ (0, 2) & (0, 3) \end{bmatrix} \end{array}$$

Mapping matrix

$$\mathbf{F} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} (2, 3) & (4, 3) \\ (0, 3) & (4, 3) \\ (3, 2) & (3, 3) \\ (7, 2) & (0, 1) \\ (0, 2) & (0, 3) \end{bmatrix} \end{array}$$

Forwarding matrix

General Case

$G = (V, E)$, n is the size of V.

A $n \times n$ (left) routing matrix $\mathbf{L}$ solves an equation of the form

$$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I},$$

over semiring S.

D is a set of destinations, with size d.

A $n \times d$ forwarding matrix is defined as

$$\mathbf{F} = \mathbf{L} \rhd \mathbf{M},$$

over some structure $(N, \Box, \rhd)$, where $\rhd \in (S \times N) \to N$.

forwarding = routing + mapping

Does this make sense?

$$\mathbf{F}(i,\ d) = (\mathbf{L} \rhd \mathbf{M})(i,\ d) = \sum_{q \in V}^{\square} \mathbf{L}(i,\ q) \rhd \mathbf{M}(q,\ d).$$

- Once again we are leaving paths implicit in the construction.
- Forwarding paths are best routing paths to egress nodes, selected with respect □-minimality.
- □-minimality can be very different from selection involved in routing.

When we are lucky ...

matrix	solves
$\mathbf{A}^*$	$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I}$
$\mathbf{A}^* \rhd \mathbf{M}$	$\mathbf{F} = (\mathbf{A} \rhd \mathbf{F}) \Box \mathbf{M}$

When does this happen?

When $(N, \Box, \rhd)$ is a (left) semi-module over the semiring S.

(left) Semi-modules

- $(S, \oplus, \otimes, \overline{0}, \overline{1})$ is a semiring.

A (left) semi-module over S

Is a structure $(N, \Box, \triangleright, \overline{0}_N)$, where

- $(N, \Box, \overline{0}_N)$ is a commutative monoid
- $\triangleright$ is a function $\triangleright \in (S \times N) \to N$
- $(a \otimes b) \triangleright m = a \triangleright (b \triangleright m)$
- $\overline{0} \triangleright m = \overline{0}_N$
- $s \triangleright \overline{0}_N = \overline{0}_N$
- $\overline{1} \triangleright m = m$

and distributivity holds,

$$
\begin{array}{lrcl}
\text{LD} : & s \triangleright (m \Box n) & = & (s \triangleright m) \Box (s \triangleright n) \\
\text{RD} : & (s \oplus t) \triangleright m & = & (s \triangleright m) \Box (t \triangleright m)
\end{array}
$$

Example : Hot-Potato

S idempotent and selective

$$\begin{array}{rcl} S & = & (S,\ \oplus_S,\ \otimes_S) \\ T & = & (T,\ \oplus_T,\ \otimes_T) \\ \rhd_{\mathrm{fst}} & \in & S \times (S \times T) \rightarrow (S \times T) \\ s_1 \rhd_{\mathrm{fst}} (s_2,\ t) & = & (s_1 \otimes_S s_2,\ t) \end{array}$$

$$\mathrm{Hot}(S,\ T) = (S \times T,\ \vec{\oplus},\ \rhd_{\mathrm{fst}}),$$

where $\vec{\oplus}$ is the (left) lexicographic product of $\oplus_S$ and $\oplus_T$.

Define $\rhd_{\mathrm{hp}}$ on matrices

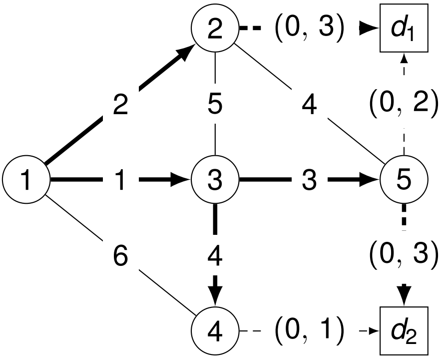
$$(\mathbf{L} \rhd_{\mathrm{hp}} \mathbf{M})(i,\ d) = \sum_{q \in V}^{\vec{\oplus}} \mathbf{L}(i,\ q) \rhd_{\mathrm{fst}} \mathbf{M}(q,\ d)$$

Sanity Check : does this implement hot-potato?

Define M to be simple if either $\mathbf{M}(v,\ d) = (1_S,\ t)$ or $\mathbf{M}(v,\ d) = (\infty_S,\ \infty_T)$.

$$
\begin{aligned}
& (\mathbf{L} \rhd_{\mathrm{hp}} \mathbf{M})(i,\ d) \\
= & \sum_{q \in V}^{\vec{\oplus}} \mathbf{L}(i,\ q) \rhd_{\mathrm{fst}} \mathbf{M}(q,\ d) \\
= & \sum_{\substack{q \in V \\ \mathbf{M}(q,\ d) = (s,\ t)}}^{\vec{\oplus}} (\mathbf{L}(i,\ q) \otimes_S s,\ t) \\
= & \sum_{\substack{q \in V \\ \mathbf{M}(q,\ d) = (1_S,\ t)}}^{\vec{\oplus}} (\mathbf{L}(i,\ q),\ t) \qquad \text{(if } M \text{ is simple)}
\end{aligned}
$$

Example of *hot-potato* forwarding



$$\mathbf{M} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \left[\begin{array}{cc} \infty & \infty \\ (0,\ 3) & \infty \\ \infty & \infty \\ \infty & (0,\ 1) \\ (0,\ 2) & (0,\ 3) \end{array}\right] \end{array}$$

Mapping matrix

$$\mathbf{F} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \left[\begin{array}{cc} (2,\ 3) & (4,\ 3) \\ (0,\ 3) & (4,\ 3) \\ (3,\ 2) & (3,\ 3) \\ (7,\ 2) & (0,\ 1) \\ (0,\ 2) & (0,\ 3) \end{array}\right] \end{array}$$

Forwarding matrix

matrix	solves
$\mathbf{A}^*$	$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I}$
$\mathbf{A}^* \rhd_{\mathrm{hp}} \mathbf{M}$	$\mathbf{F} = (\mathbf{A} \rhd_{\mathrm{hp}} \mathbf{F}) \vec{\oplus} \mathbf{M}$

Example : Cold-Potato

T idempotent and selective

$$\begin{array}{rcl} S & = & (S,\ \oplus_S,\ \otimes_S) \\ T & = & (T,\ \oplus_T,\ \otimes_T) \\ \rhd_{\text{fst}} & \in & S \times (S \times T) \to (S \times T) \\ s_1 \rhd_{\text{fst}} (s_2,\ t) & = & (s_1 \otimes_S s_2,\ t) \end{array}$$

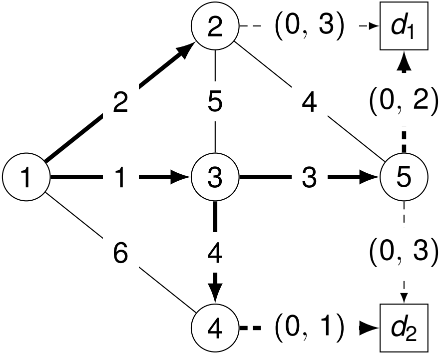
$$\text{Cold}(S,\ T) = (S \times T,\ \overleftarrow{\oplus},\ \rhd_{\text{fst}}),$$

where $\vec{\oplus}$ is the (left) lexicographic product of $\oplus_S$ and $\oplus_T$.

Define $\rhd_{\text{cp}}$ on matrices

$$(\mathbf{L} \rhd_{\text{cp}} \mathbf{M})(i,\ d) = \sum_{q \in V}^{\overleftarrow{\oplus}} \mathbf{L}(i,\ q) \rhd_{\text{fst}} \mathbf{M}(q,\ d)$$

Example of *cold-potato* forwarding



$$\mathbf{M} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} \infty & \infty \\ (0,\ 3) & \infty \\ \infty & \infty \\ \infty & (0,\ 1) \\ (0,\ 2) & (0,\ 3) \end{bmatrix} \end{array}$$

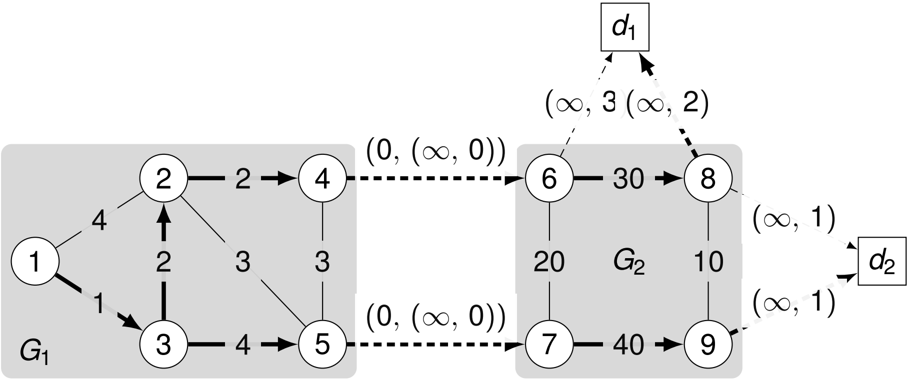
Mapping matrix

$$\mathbf{F} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} (4,\ 2) & (5,\ 1) \\ (4,\ 2) & (9,\ 1) \\ (3,\ 2) & (4,\ 1) \\ (7,\ 2) & (0,\ 1) \\ (0,\ 2) & (7,\ 1) \end{bmatrix} \end{array}$$

Forwarding matrix

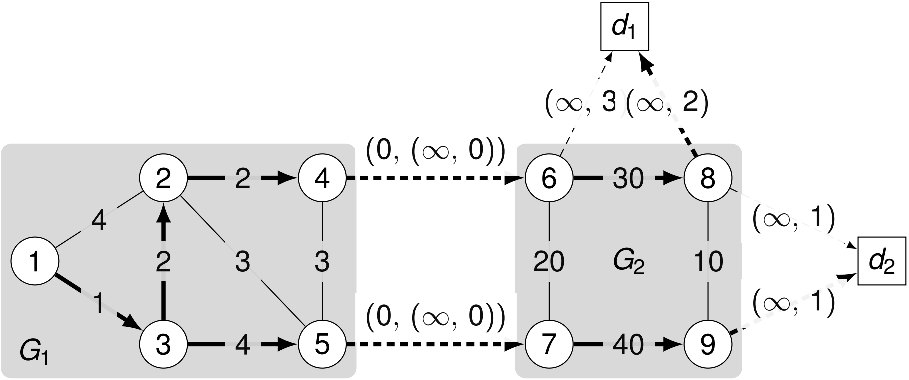
matrix	solves
$\mathbf{A}^*$	$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I}$
$\mathbf{A}^* \rhd_{\mathrm{cp}} \mathbf{M}$	$\mathbf{F} = \mathbf{A} \rhd_{\mathrm{cp}} \mathbf{F} \overleftarrow{\oplus} \mathbf{M}$

A simple example of route redistribution



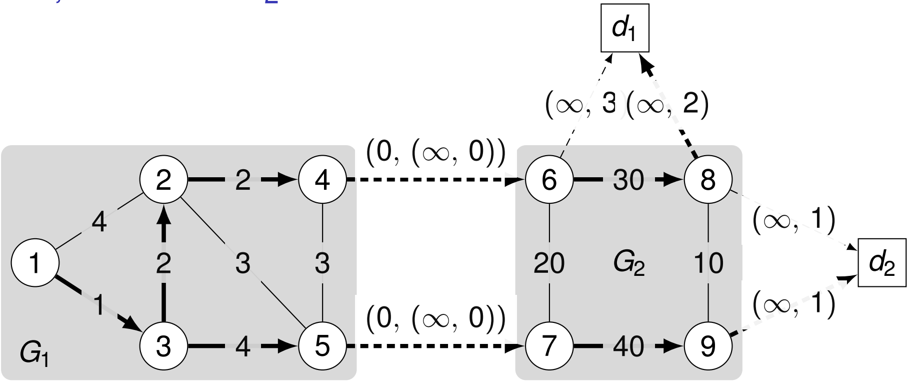
We will will use the routing and mapping of G_2 to construct a forwarding $\mathbf{F}_2$, that will be passed as a mapping to G_1 ...

A simple example of route redistribution



- G_2 is routing with the bandwidth semiring bw
- G_2 is forwarding with $\mathrm{Cold}(\mathrm{bw},\ \mathrm{sp})$
- G_1 is routing with the bandwidth semiring sp
- G_1 is forwarding with $\mathrm{Hot}(\mathrm{sp},\ \mathrm{Cold}(\mathrm{bw},\ \mathrm{sp}))$

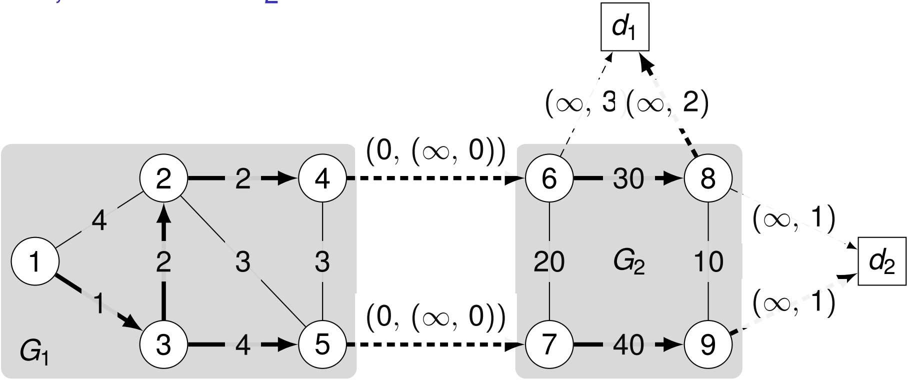
First, construct $\mathbf{F}_2$



$$\mathbf{L}_2 = \begin{array}{c} \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccc} 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} \infty & 20 & 30 & 20 \\ 20 & \infty & 20 & 40 \\ 30 & 20 & \infty & 20 \\ 20 & 40 & 20 & \infty \end{bmatrix} \end{array}$$

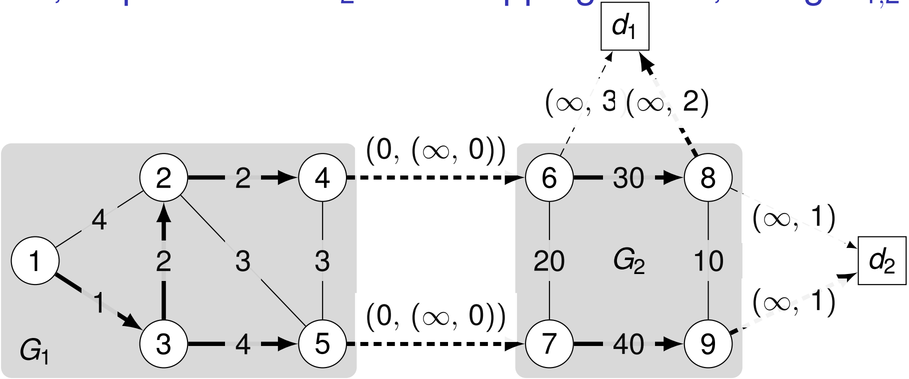
$$\mathbf{M}_2 = \begin{array}{c} \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{bmatrix} (\infty, 3) & \infty \\ \infty & \infty \\ (\infty, 2) & (\infty, 1) \\ \infty & (\infty, 1) \end{bmatrix} \end{array}$$

First, construct $\mathbf{F}_2$



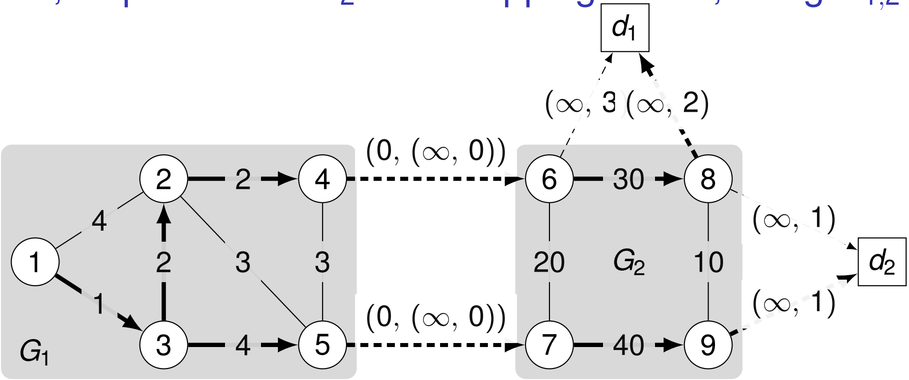
$$\mathbf{F}_2 = \mathbf{L}_2 \rhd_{\mathrm{cp}} \mathbf{M}_2 = \begin{array}{c} \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \left[\begin{array}{cc} (30, 2) & (30, 1) \\ (20, 2) & (40, 1) \\ (\infty, 2) & (\infty, 1) \\ (20, 2) & (\infty, 1) \end{array}\right] \end{array}$$

Now, ship it over to G_2 as a mapping matrix, using $\mathbf{B}_{1,2}$



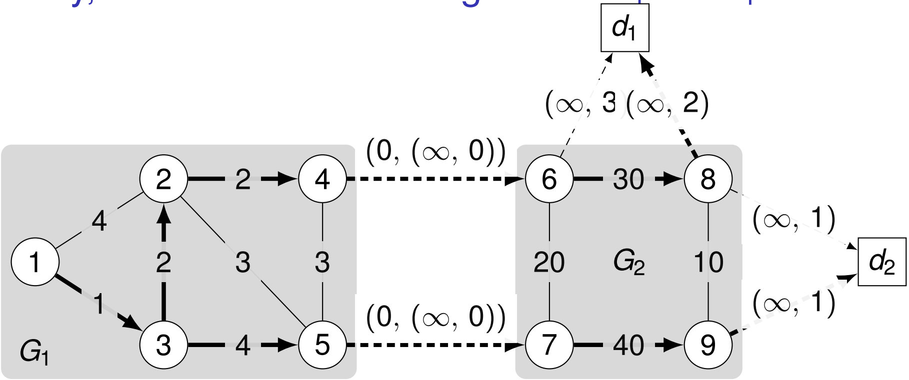
$$
\mathbf{B}_{1,2} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c} \begin{array}{cccc} 6 & 7 & 8 & 9 \end{array} \\
\begin{bmatrix}
\infty & \infty & \infty & \infty \\
\infty & \infty & \infty & \infty \\
\infty & \infty & \infty & \infty \\
(0, (\infty, 0)) & \infty & \infty & \infty \\
\infty & (0, (\infty, 0)) & \infty & \infty
\end{bmatrix} \end{array}
$$

Now, ship it over to G_2 as a mapping matrix, using $\mathbf{B}_{1,2}$



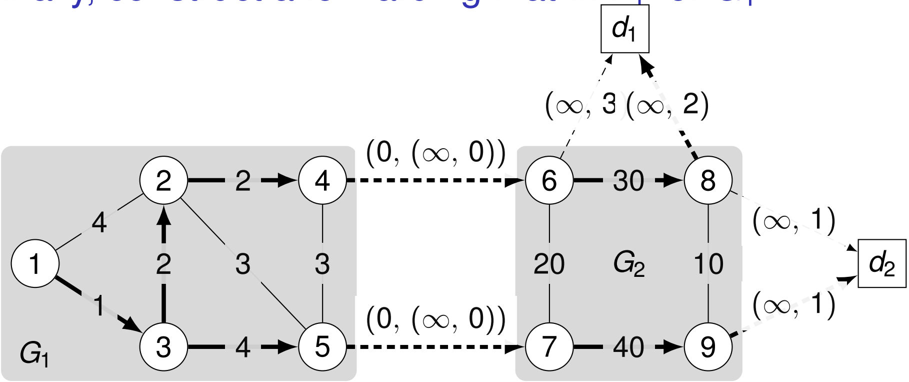
$$\mathbf{M}_1 = \mathbf{B}_{1,2} \lhd_{\text{hp}} \mathbf{F}_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \left[\begin{array}{cc} \infty & \infty \\ \infty & \infty \\ \infty & \infty \\ (0, (30, 2)) & (0, (30, 1)) \\ (0, (20, 2)) & (0, (40, 1)) \end{array} \right] \end{array}$$

Finally, construct a forwarding matrix $\mathbf{F}_1$ for G_1



$$
\mathbf{L}_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\left[\begin{array}{ccccc}
0 & 3 & 1 & 5 & 5 \\
3 & 0 & 2 & 2 & 3 \\
1 & 2 & 0 & 4 & 4 \\
5 & 2 & 4 & 0 & 3 \\
5 & 3 & 4 & 3 & 0
\end{array}\right] \end{array}
$$

Finally, construct a forwarding matrix $\mathbf{F}_1$ for G_1



$$\mathbf{F}_1 = \mathbf{L}_1 \rhd_{\mathrm{hp}} \mathbf{M}_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \left[\begin{array}{cc} (5, (30, 2)) & (5, (40, 1)) \\ (2, (30, 2)) & (2, (30, 1)) \\ (4, (30, 2)) & (4, (40, 1)) \\ (0, (30, 2)) & (0, (30, 1)) \\ (0, (20, 2)) & (0, (40, 1)) \end{array} \right] \end{array}$$

Bibliography I

[BG09] John N. Billings and Timothy G. Griffin.
A model of internet routing using semi-modules.
In *11th International Conference on Relational Methods in Computer Science (RelMiCS10)*, November 2009.

[Car79] Bernard Carré.
Graphs and Networks.
Oxford University Press, 1979.

[Con71] J. H. Conway.
Regular Algebra and Finite Machines.
Chapman and Hall, 1971.

[GM08] M. Gondran and M. Minoux.
Graphs, Dioids, and Semirings : New Models and Algorithms.
Springer, 2008.

Bibliography II

[Gur08] Alexander Gurney.
Designing routing algebras with meta-languages.
Thesis in progress, 2008.